Lecture 9: Dispersion, resonance, phase & group velocities.

Remember how I said the dielectric constant ε isn't constant?
Time to deal with that.

Most prominently, ε is a function of frequency, $\varepsilon = \varepsilon(\omega)$

And that means $n = n(\omega)$, so among other things, different colors of light refract differently, and materials like glass <u>disperse</u> white light into a variety of colors.

We're going to look at a classical model of dispersion, starting from the idea that time-dependent E-fields drive electrons about a nucleus. Their motion classically is given by:

$$m\ddot{\vec{x}} = -K\vec{x} - \gamma\dot{\vec{x}} - e\vec{E}_0 e^{-i\omega t}$$

↑ restoring force   ↑ Damping   ↖ driving

A good old fashioned damped, driven SHO

(Clicker question re: damping)

In theory you know how to solve this and get:

(1) $$\vec{x}(t) = \frac{-e\vec{E}_0}{K - m\omega^2 - i\omega\gamma} e^{-i\omega t}$$ for steady-state, which is what we care about.

Somehow we'd like to get from this to n as a function of ω. We know $n = c/v$ and $v = \frac{1}{\sqrt{\varepsilon\mu}}$, so getting ε would give us an angle here.

This is how we do that:

The definition of a dipole moment is: $\vec{p} = q\vec{d}$ or $-e\vec{x} = \alpha\vec{E}$

with that last bit being true as long as we're in a linear regime

So looking at (1) we see $\vec{x} = -\frac{\alpha}{e}\vec{E}$

$$\Rightarrow \quad \alpha = \frac{e^2}{K - m\omega^2 - i\omega\gamma}$$

$\vec{p}$ is a dipole moment, and the polarization $\vec{P}$ is a dipole moment per volume, so if we have an atomic number density ν,

$$\vec{P} = \nu \vec{p} = \nu \alpha \vec{E}$$

Also,
$$\vec{D} = \varepsilon \vec{E} = \varepsilon_0 \vec{E} + \vec{P} = \varepsilon_0 \vec{E} + \nu \alpha \vec{E}$$

So $\quad \varepsilon = \varepsilon_0 + \nu \alpha \quad$ and $\quad \varepsilon(\omega) = \varepsilon_0 + \dfrac{\nu e^2}{K - m\omega^2 - i\omega\gamma}$

That means ε is complex. Will that make the index of refraction complex? We can avoid that.

Let's write $\quad K = \omega/v = \omega\sqrt{\varepsilon\mu} \quad$ (usually $\mu = \mu_0$)

That'll make K complex, so $\quad K = K_1 + iK_2$

$$\Rightarrow \quad \vec{E}(t) = \vec{E}_0 e^{i(Kx - \omega t)} = \vec{E}_0 e^{-K_2 x} e^{i(K_1 x - \omega t)}$$

So the complex part of ε & K have physical interpretations: they're associated with attenuation, or absorption.

We can pull that out and say that the actual wave speed is associated with the real part of K, so it's more appropriate to say
$$v = \omega/K_1, \quad \text{so} \quad n = \frac{cK_1}{\omega} \quad \text{and is real.}$$

We can go back through and do a little algebra and an expansion to get:

$$n(\omega) = 1 + \frac{\nu e^2}{2\varepsilon_0 m \omega_0^2} \frac{\omega_0^2(\omega_0^2 - \omega^2)}{(\omega_0^2 - \omega^2)^2 + (\omega\gamma/m)^2} \qquad \text{With } \omega_0^2 \equiv K/m$$

We can even get $d \equiv 1/2K_2$, a characteristic absorption length, but I won't bother to write it out.

Question: Why the 2?

(graph)

So what we have here is a _resonance_: Absorption gets really strong around ω_{01} and $n(\omega)$ gets wild also.

Real materials usually have many resonances, although there also tend to be large frequency bands with no resonances and a monotonic $n(\omega)$

(graphs)

Phase & Group velocity   (clicker question)

I keep saying $v = \omega/k$, but in localized waves (as opposed to plane waves of infinite extent) there are actually _two_ things we can call velocities.

There's the group velocity, the velocity of an overall envelope, and the phase velocity, the velocity of the wiggles under the envelope (applet)

Only the group velocity has to obey $v < c$. The phase velocity $> c$ with some frequency.

$v_p = \omega/k$ is the phase velocity, which can be above c
    Quick visualization: $kx - \omega t = $ constant

$v_g = d\omega/dk$ is the group velocity, which is always $< c$