MATH 225 - June 11, 2008 NAME: Key
Quiz IV -

In order to receive full credit, SHOW ALL YOUR WORK. Full credit will be given only if all reasoning and work is provided. When applicable, please enclose your final answers in boxes.

1. (2 Points) Using the integral definition, calculate the Laplace transform of $f_1(t) = \sin(at)$ and $f_2(t) = \cos(at)$, $a \in \mathbb{R}$.

$$\mathcal{L}\{e^{iat}\} = \int_0^\infty e^{iat} e^{-st}\,dt = \frac{1}{ia-s} e^{(ia-s)t}\Big|_0^\infty = \frac{1}{s-ia}\cdot\frac{s+ia}{s+ia} = \frac{s+ia}{s^2+a^2}$$

Note:

$$\text{Real}\{\mathcal{L}\{e^{iat}\}\} = \mathcal{L}\{f_2(t)\} = \frac{s}{s^2+a^2}$$

$$\text{Img}[\mathcal{L}\{e^{iat}\}] = \mathcal{L}\{f_1(t)\} = \frac{a}{s^2+a^2}$$

2. (2 Points) Solve the following initial value problems:

(a) $y'' - y = 0, \quad y(0) = -1,\ y'(0) = 1$

$$\mathcal{L}\{y''-y\} = \mathcal{L}\{y''\} - \mathcal{L}\{y\} = s^2Y(s) - s\,\overset{-1}{\cancel{y(0)}} - \overset{1}{\cancel{y'(0)}} - Y(s) = \mathcal{L}\{0\} = 0$$

$$\Rightarrow \quad Y(s) = \frac{1-s}{s^2-1} = \frac{A}{s-1} + \frac{B}{s+1} = \frac{A(s+1)+B(s-1)}{s^2-1} \Rightarrow$$

$s=-1$: $2 = B(-2)$, $B=-1$

$s=1$: $\Rightarrow A=0$ Oh, yeah, I should have noticed

$$= \frac{-1}{s+1}$$

$$\Rightarrow \mathcal{L}^{-1}\{Y(s)\} = y(t) = -e^{-t}$$

(b) $y'' + y = 0, \quad y(0) = -1,\ y'(0) = 1$

$$\mathcal{L}\{y''+y\} = 0 \Rightarrow \quad Y(s) = \frac{1-s}{s^2+1} = \frac{-s}{s^2+1} + \frac{1}{s^2+1}$$

$$\Rightarrow \mathcal{L}^{-1}\{Y(s)\} = y(t) = -\cos(t) + \sin(t)$$

(c) $y'' = 0, \quad y(0) = -1,\ y'(0) = 1$

$$\mathcal{L}\{y''\} = 0 \Rightarrow Y(s) = \frac{1-s}{s^2} = \frac{1}{s^2} - \frac{1}{s} \Rightarrow \mathcal{L}\{Y(s)\} = t - 1 = y(t)$$

3. (2 Points) Solve the initial value problem $y'' + 2y' + 9y = 1 - u_2(t) + \delta_3(t)$, $\quad y(0) = 0,\ y'(0) = 1$.

$$Y(s) = \frac{1}{s^2+2s+9} - \frac{1}{s^2+2s+9}\left\{ \frac{-e^{-2s}}{s} + e^{-3s} + \frac{1}{s} \right\} =$$

Note

$$\frac{1}{s}\cdot\frac{1}{s^2+2s+9} = \frac{A}{s} + \frac{Bs+C}{s^2+2s+9} = \frac{A(s^2+2s+9)+Bs^2+Cs}{s(s^2+2s+9)}$$

$\Rightarrow s=0$

$A = \frac{1}{9}$, $B = -\frac{1}{9}$, $2A = -C \Rightarrow C = -\frac{2}{9}$

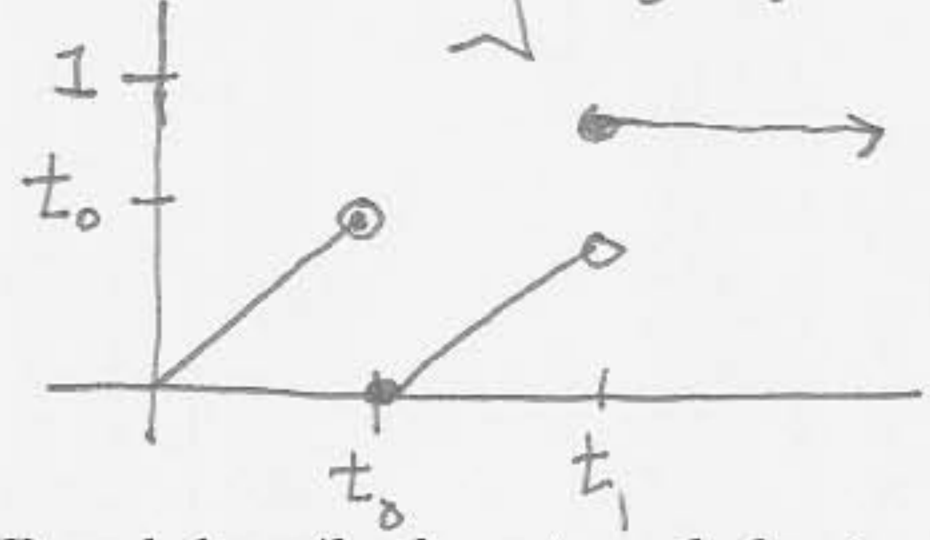
$$= \underbrace{\frac{1}{s^2+2s+9}}_{(s+1)^2+8} - \frac{e^{-3s}}{s^2+2s+9} + \left[\frac{1}{9s} - \frac{1}{9}\frac{s+2}{s^2+2s+9}\right]\cdot\left\{1 - e^{-2s}\right\}$$

$$= \frac{1}{2\sqrt{2}}\frac{2\sqrt{2}}{(s+1)^2+8} - \frac{1}{2\sqrt{2}}e^{-3s}\cdot\frac{1}{(s+1)^2+8} + \frac{1}{9s} - \frac{e^{-2s}}{9s} - \frac{1}{9}\left(\frac{s+1}{(s+1)^2+8} + \frac{2\sqrt{2}}{2\sqrt{2}[(s+1)^2+8]}\right) \Rightarrow \cancel{y(t) = \frac{1}{2\sqrt{2}} e^{-t}\sin(2\sqrt{2}t) - \frac{1}{2\sqrt{2}}u}$$

4. (2 Points) Sketch a graph of $f(t)$ where $f(t) = t - u_{t_0}(t)t + u_{t_0}(t)(t - t_0) - u_{t_1}(t)(t - t_0) + u_{t_1}(t)$.

$$= t\left(u_0(t) - u_{t_0}(t)\right) + (t - t_0)\left(u_{t_0}(t) - u_{t_1}(t)\right) + u_{t_1}(t)$$

Assuming $t_0 < 1$



5. (2 Points) Solve the following IVP and describe long-term behavior of the oscillations.

$$\frac{d^2y}{dt^2} + y = \sum_{n=0}^{\infty} \delta_n(t), \quad y(0) = 0,\ y'(0) = 0. \tag{1}$$

$$Y(s) = \frac{1}{s^2+1}\sum_{n=0}^{\infty} e^{-ns} \Rightarrow y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s+1}\sum_{n=0}^{\infty} e^{-ns}\right\} =$$

$$= \sum_{n=0}^{\infty} u_n(t)\sin(t-n)$$

$$y(t) = \frac{1}{2\sqrt{2}}\cdot e^{-t}\sin(2\sqrt{2}t) - \frac{1}{2\sqrt{2}}u_3(t)e^{-t+3}\sin(2\sqrt{2}(t-3)) + \frac{1}{9} - u_2(t) -$$

$$- \frac{1}{9}\left(e^{-t}\cos(2\sqrt{2}t) + \frac{1}{2\sqrt{2}}e^{-t}\sin(2\sqrt{2}t)\right) + \frac{u_2(t)e^{-t+2}}{(2\sqrt{2})9}\sin(2\sqrt{2}(t-2)) + \frac{u_2(t)}{2\sqrt{2}}e^{-t+2}\cos(2\sqrt{2}(t-2))$$