Filamentation - modulation instability
with smooth beam profile $\rightarrow P_{cr}$
but any modulation on the profile can grow $\rightarrow$ beam breakup.

Angular spectrum of a beam:
transverse field profile at $z=0$ be
$U(x,y)$
consider Fourier transform
$$A(f_x, f_y) \equiv \iint U(x,y)\, e^{-i2\pi(f_x x + f_y y)}\, dx\, dy$$
$$= \mathcal{F}\{U(x,y)\}$$

$A(\ )$ is the angular spectrum
f_x, f_y are spatial frequencies
consider a plane wave
$$E_p = \exp(i\vec{k}\cdot\vec{r}) = \exp(i(k_x x + k_y y + k_z z))$$
$$A_p = \int e^{ik_x x} e^{-i2\pi f_x x}\, dx \int e^{ik_y y} e^{-i2\pi f_y y}\, dy$$

in spatial domain,
$$\mathcal{F}\{e^{\pm i 2\pi f_{x0} x}\} \equiv \delta(f_x \pm f_{x0})$$
here, $f_{x0} = k_x/2\pi = \frac{k_0}{2\pi}\sin\theta_x = \frac{1}{\lambda}\sin\theta_x$

$\therefore A_p(f_x, f_y) = \delta(f_x - \frac{1}{\lambda}\sin\theta_x)\,\delta(f_y - \frac{1}{\lambda}\sin\theta_y)$
each spatial frequency is a plane wave traveling at an angle.
"angular spectrum"

NL analysis:

consider $E(\vec{r},t) = E(\vec{r})e^{-i\omega t} + c.c.$

here $E(r)$ has 3 plane wave components:

$$E(\vec{r}) = E_0(\vec{r}) + E_{+1}(\vec{r}) + E_{-1}(\vec{r})$$

$$= \underset{\text{strong}}{A_0(z)e^{ik_0 z}} + a_{+1}(z)e^{i(+\vec{q}\cdot\vec{r} + k_0 z)} + a_{-1}(z)e^{i(-\vec{q}\cdot\vec{r}+k_0 z)}$$

weak

we will show that the power in the weak side bands can grow, taking power from the main beam:

momentum transfer:

wave eqn:

$$2ik\frac{\partial A}{\partial z} + \nabla_T^2 A = -\frac{4\pi\omega^2}{c^2}P_{NL}$$

$$P_{NL} = 3\chi^{(3)}|E|^2E \equiv P_0 + P_{+1} + P_{-1} = (p_0 + p_{+1} + p_{-1})e^{\phi}$$

$\rightarrow$ driving term at each value of k_0, k_{+1}, k_{-1}

have 3 separate, coupled eqns.

$|E|^2E = |E_0|^2E_0 + |E_1|^2E_1 + |E_{-1}|^2E_{-1}$ + many cross terms.

look at exponentials:

$|E_0|^2E_0 \rightarrow e^{ik_0 z}$ in 0 dir.

$|E_1|^2E_1 \rightarrow e^{i(\vec{q}\cdot\vec{r}+k_0 z)}$ in +1 dir

$E_1^2E_0^* \rightarrow e^{i(2\vec{q}\cdot\vec{r}+k_0 z)}$ in +2 dir ignore this for now.

$E_0^2E_1^* \rightarrow e^{i(3k_0 z - q\cdot r)}$

Keep only terms that drive signal in the $0, \pm 1$ directions.

$$P_0 = 3\chi^{(3)}\left(|A_0|^2 + \cancel{|A_1|^2 + |A_{-1}|^2}\right)A_0 e^{ikz}$$

small: assume no back-coupling

$$= p_0 e^{ikz}$$

$$2ik\frac{\partial A_0}{\partial z} + \nabla_\perp^2 A_0 = -\frac{4\pi\omega^2}{c^2}p_0$$

solution is like: $A_0(z) = A_{00} e^{i\gamma z}$

$$2\gamma k = \frac{4\pi\omega^2}{c^2}\cdot 3\chi^{(3)}|A_{00}|^2 A_0 \longrightarrow \gamma = \frac{6\pi\omega\chi^{(3)}|A_{00}|^2}{n_0 c} = n_2 k_{vac} I$$

so that $\gamma z = B =$ NL phase.

A_{00} = initial value, non-depleted, real

In this approx, central beam sees a NL phase shift, but negligible loss.

$$P_{\pm 1} = 3\chi^{(3)}\left(2|A_0|^2 A_{\pm 1} + A_0^2 A_{\mp 1}^*\right)e^{ikz} = p_\pm e^{ikz}$$

dropping $|A_1|^2, |A_{-1}|^2$ terms.

with soln to $A_0(z)$, $p_\pm = 3\chi^{(3)}\left(2|A_{00}|^2 A_{\pm 1} + A_{00}^2 e^{2i\gamma z} A_{\mp 1}^*\right)$

now calc. growth of $A_{\pm 1}$

$$2ik\frac{\partial A_{\pm 1}}{\partial z} + \nabla_\perp^2 A_{\pm 1} = -\frac{4\pi\omega^2}{c^2}p_{\pm 1}$$

let $A_{\pm 1} = a_{\pm 1} e^{\pm i\vec{q}\cdot\vec{r}}$ $\vec{q}$ is in transverse direction

$$2ik\frac{da_{\pm 1}}{dz} - q^2 a_{\pm 1} = -\frac{4\pi\omega^2}{c^2}\cdot 3\chi^{(3)}|A_{00}|^2\left(2a_{\pm 1} + a_{\mp 1}^* e^{2i\gamma z}\right)$$

use γ as defined above

$$\frac{da_{\pm 1}}{dz} + \frac{iq^2}{2k}a_{\pm 1} = i\gamma\left(2a_{\pm 1} + a_{\mp 1}^* e^{2i\gamma z}\right)$$

absorb phase into amplitude factor, defining $a_{\pm 1} = a'_{\pm 1} e^{i\gamma z}$

$$\left(\frac{da'_{\pm}}{dz}\right)\cancel{e^{+i\gamma z}} + i\gamma a'_{\pm 1}\cancel{e^{+i\gamma z}} + \frac{iq^2}{2k}a'_{\pm}\cancel{e^{+i\gamma z}} = i\gamma\left(2a'_{\pm 1}\cancel{e^{+i\gamma z}} + a'^*_{\mp 1}\cancel{e^{i\gamma z}}\right)$$

Now $\frac{d}{dz} a'_{\pm 1} = i(\gamma - q^2/2k)\, a'_{\pm 1} + i\gamma\, a'^{*}_{\mp 1}$

with $\beta = q^2/2k$, we write the two equations in matrix form:

$$\frac{d}{dz}\begin{pmatrix} a'_1 \\ a'^{*}_{-1} \end{pmatrix} = \begin{pmatrix} i(\gamma-\beta) & i\gamma \\ -i\gamma & -i(\gamma-\beta) \end{pmatrix} \begin{pmatrix} a'_1 \\ a'^{*}_{-1} \end{pmatrix}$$

assume solutions of form $a'_1(0)\, e^{\Lambda z}$

→ eigenvalue eqn → gain:

$$\Lambda = \pm\sqrt{\beta(2\gamma-\beta)}$$

actual gain (Λ real) if $\gamma > \frac{1}{2}\beta$ or $q_{max} = 2\sqrt{k\gamma}$

→ $n_2 > 0$

given γ → gain depends on q: peak at $q_{opt} = q_{max}/\sqrt{2}$

Now, q is the transverse wavenumber, like k_x

$\theta_{opt} \sim q_{opt}/k =$

Phase conjugation

A phase conjugate wave is the conjugate of the spatial part of the wave:

$$\vec{E}(r,t) = \vec{E}(r)\, e^{-i\omega t} + c.c.$$

phase conj:

$$\vec{E}_c(r,t) \to \vec{E}^*(r)\, e^{-i\omega t} + c.c.$$

A phase conjugate mirror (PCM) will generate this wave.

if

$$\vec{E}(r) = \underbrace{\hat{\varepsilon}}_{\text{polarization}}\ \underbrace{a(\vec{r})}_{\text{ampl. profile}}\ \underbrace{e^{i\phi(r)}}_{\text{wavefront}}\ \underbrace{e^{i\vec{k}\cdot\vec{r}}}_{\text{plane wave}}\ e^{-i\omega t}$$

then

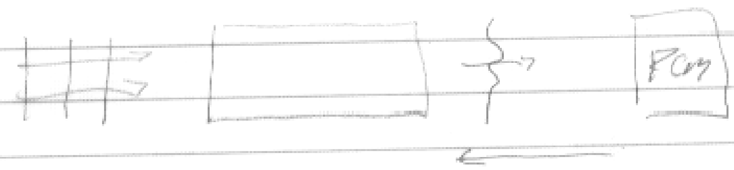
$$\vec{E}_c(r) = \underbrace{\hat{\varepsilon}^*}_{\text{polariz}}\ a(r)\ \underbrace{e^{-i\phi(r)}}_{\text{rev. wavefront}}\ \underbrace{e^{-i\vec{k}\cdot\vec{r}}}_{\text{reversed direction}}\ e^{-i\omega t}$$

PC is like time-reversal.

LH goes to LH

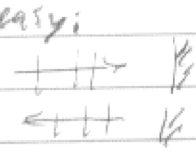
normal mirror:

PCM can be used to remove aberrations on a double-pass.



Some phase conj. waves are easy:

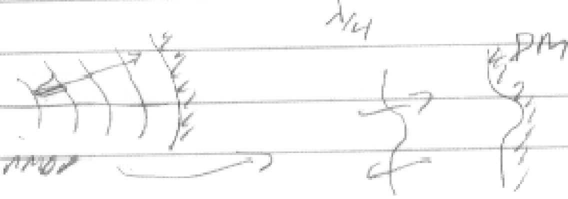
plane wave: linear polariz.



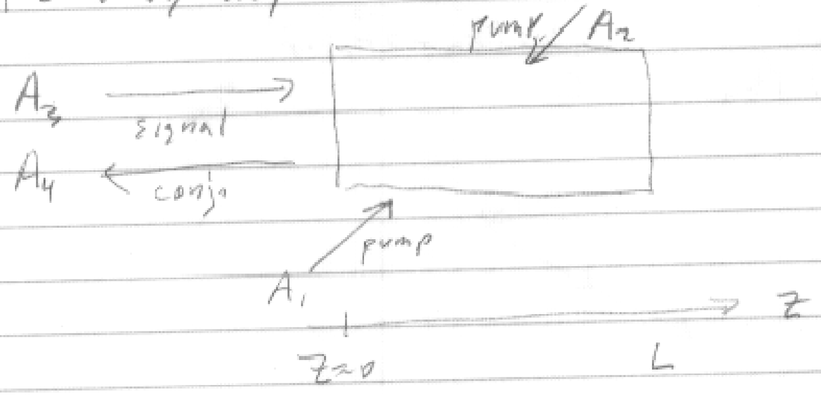
for circ polariz would need $\lambda/4$

spherical wave

on deformable mirror



PCM by degenerate four wave mixing (DFWM)



all waves have same ω (degenerate)

$A_2 = A_1^*$ typically plane waves, stronger waves than signal

$P^{NL} = 3\chi^{(3)}(\vec{E}\cdot\vec{E}^*)\vec{E}$ $\quad E = E_1 + E_2 + E_3$

many terms: all at ω
different output $\vec{k}$'s

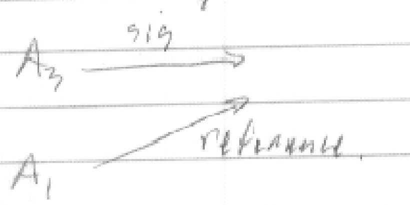
$$P^{NL} = 6\chi^{(3)} A_1 A_2 A_3^* \exp\left(i(\vec{k}_1 + \vec{k}_2 - \vec{k}_3)\cdot\vec{r}\right)$$

Arrange that $\vec{k}_1 = -\vec{k}_2$ and $A_1 = A_2^*$

$$\rightarrow P^{NL} = 6\chi^{(3)} |A_1|^2 A_3^* e^{-i\vec{k}_3\cdot\vec{r}}$$

on RHS of wave eqn. for E_4, this is a source for the conjugate wave.

A PCM is a dynamic hologram:



$$|A_1 + A_3|^2 = |A_1|^2 + |A_3|^2 + A_1^* A_3 + A_1 A_3^*$$

interference.

through the NL index this interference produces a (volume) phase grating in medium

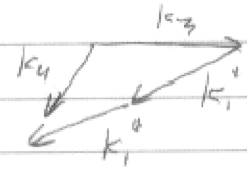
Now add readout beam: A_2

the term $A_1 A_3^* A_2 \rightarrow$ conj. wave

Note that a wave like $A_1^* A_3 A_2$, with $A_2 = A_1^*$
leads to $e^{i(\vec{k}_3 - 2\vec{k}_1)\cdot\vec{r}}$ as driving term for A_4

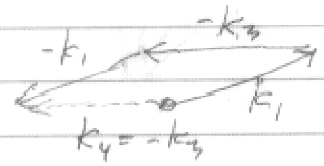
$$\nabla^2 A_4 + \frac{\varepsilon \omega^2}{c^2} A_4 = -\frac{4\pi\omega^2}{c^2} (A_1^*)^2 A_3 \, (6\chi^{(3)}) \, e^{i(\vec{k}_3 - 2\vec{k}_1)\cdot\vec{r}}$$

orient coord. system along $\vec{k}_3 - 2\vec{k}_1$:



→ inherent phase mismatch, vectors determine direction of wave, but Δk controls yield

for PC wave,
$A_1 A_3^* A_2 \rightarrow k_1 + k_2 - k_3$ w/ $k_2 = -k_1$



this is perfectly phase matched.

DFWM analysis
$E_{tot} = E_1 + E_2 + E_3 + E_4$

calc $P^{NL} = 3\chi^{(3)} E^2 E^*$
→ many terms.
keep track of k's: sum → output direction.

ex: $P_4 = 3\chi^{(3)} \Big(\underbrace{E_4^2 E_4^*}_{SPM} + \underbrace{2 E_4 \left(|E_1|^2 + |E_2|^2 + |E_3|^2\right)}_{XPM} + \underbrace{2 E_1 E_2 E_3^*}_{mix.} \Big)$

last term is what we want.
We chose to keep that one by construction
b/c $E_1 = E_2^*$ and $E_4 \to E_3^*$

Method: E_1, E_2 strong E_3, E_4 weak.
→ keep only terms w/ one E_3 or E_4
then solve E_1, E_2 independent
→ E_3, E_4

Solve for A_1: $E_1 = A_1 e^{i(\vec{k}_1 \cdot \vec{r} - wt)} + c.c.$

$$\nabla^2 A_1 + \frac{\varepsilon w^2}{c^2} A_1 = -\frac{4\pi w^2}{c^2} P_1$$

$$w/\ P_1 = 3\chi^{(3)} \left(A_1 \left(|A_1|^2 + 2|A_2|^2 \right) \right)$$

orient coord system along $\vec{k}_1$ (z' direction)

Slowly varying approx

$$\rightarrow \frac{dA_1}{dz} = i \frac{6\pi w}{nc} \chi^{(3)} \left(|A_1|^2 + 2|A_2|^2 \right) A_1 \equiv i K_1 A_1$$

$$A_1(z') = A_1(0) e^{iK_1 z}$$

A_1 picks up a phase modulation from itself + $|A_2|^2$

- note that K_1 is like an additional component to k_1

for A_2,

$$\frac{dA_2}{dz} = -i \frac{6\pi w}{nc} \chi^{(3)} \left(|A_2|^2 + 2|A_1|^2 \right) A_2 \equiv -i K_2 A_2$$

we're assuming non-depletion of A_1, A_2 $\rightarrow K_1 K_2$ are constant

note that if $|A_1| = |A_2|$, $K_1 = K_2$

In expr for A_3, A_4 we have product $A_1 A_2$ in driving term

K's change length of k's

$$\rightarrow A_1(0) A_2(0) \underbrace{e^{i(K_1 - K_2)z'}}_{\rightarrow 1 \text{ for } |A_1| = |A_2|} \rightarrow \text{phase mismatch.}$$

Now put this into A_3', A_4'

$$\frac{dA_3}{dz} = \frac{12\pi i w}{nc} \chi^{(3)} \left[\underbrace{\left(|A_1|^2 + |A_2|^2 \right)}_{XPM} A_3 + A_1 A_2 A_4^* \right]$$

$$\equiv i K_3 A_3 + i K A_4^*$$

Absorb K_3 into A_3, $A_3 = A_3' e^{iK_3 z}$ $\quad k_3' = k_3 + K_3$

$A_4 = A_4' e^{-iK_4 z}$ $\quad k_4' = -k_3 - K_3$

$\rightarrow \dfrac{dA_3'}{dz} = i\kappa A_4'^{*}$ $\qquad \dfrac{dA_4'}{dz} = -i\kappa A_3'^{*}$

A_4' driven by $A_3'^{*}$

put into one eqn:

$$\frac{d^2}{dz^2} A_4' + |\kappa|^2 A_4' = 0$$

B.C. $A_3(0) = A_{30}$

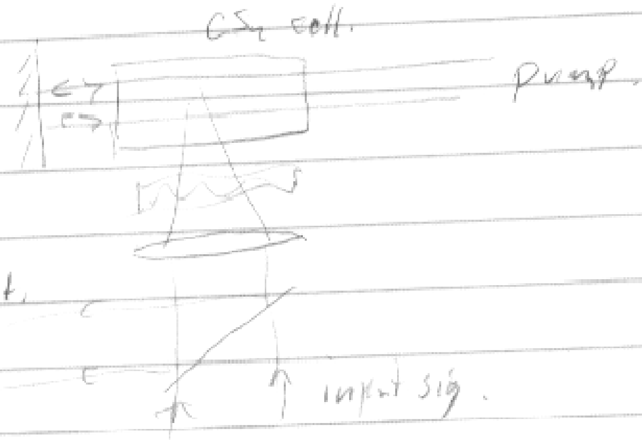
$A_4(L) = 0$

$\rightarrow A_3'^{*}(L) = \dfrac{A_3'^{*}(0)}{\cos|\kappa|L}$ grows

$A_4'(0) = \dfrac{i\kappa}{|\kappa|}\left(\tan|\kappa|L\right) A_3'^{*}(0)$ also grows.

reflectivity > 100% b/c of energy from pump beams.

setup:



Vector PCM

must have cancellation of phase of pump beams.

$\rightarrow$ req't that they be counter-rotating.