Day 23: The vector potential

Since $\vec{B}$ has curl, we <u>cannot</u> write it as the gradient of a <u>scalar</u> function. But this leaves us with other options.

Earlier we derived that $\vec{B}(\vec{x}) = \vec{\nabla}\times\left[\frac{\mu_0}{4\pi}\int\frac{J(\vec{x}')\,d^3x'}{|\vec{x}-\vec{x}'|}\right]$

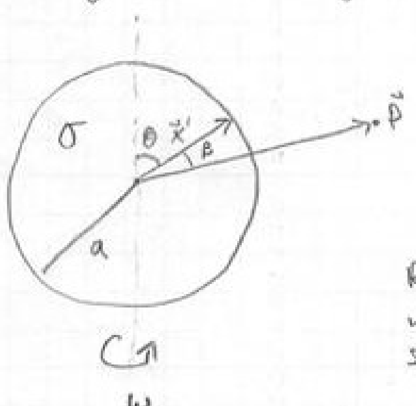
If we define the term in brackets to be $\vec{A}$, then $\vec{B} = \vec{\nabla}\times\vec{A}$.

$\vec{E} = -\nabla V$ says $\vec{E}$ can be recovered by taking the derivative of a scalar function, the potential

$\vec{B} = \nabla\times\vec{A}$ says $\vec{B}$ can be recovered by taking the derivative (the curl is a kind of derivative) of a <u>vector</u> function. $\vec{A}$ is the vector potential.

$\vec{A}$ is not quite as advantageous as V. Being scalar, V is usually not hard to calculate. $\vec{A}$ is sometimes just as hard to calculate as $\vec{B}$, but nevertheless we will find applications for it frequently, especially as we move to unify electricity and magnetism.

Finding $\vec{A}$: A rotating, hollow charged sphere



This is a fairly simple physical configuration, but finding $\vec{A}$ will not be easy. We start from:

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int\frac{\vec{K}(\vec{x}')\,dA'}{|\vec{x}-\vec{x}'|}$$

Remember: $\vec{x}$ points to the place at which we want to know $\vec{A}$. $\vec{x}'$ points to a little chunk of source.

(clicker question)

Ok, moving right along, $\vec{K} = \sigma\vec{v}$ and $\vec{v} = w\vec{r} = wa\sin\theta'\hat{\phi}$

$$\Rightarrow \quad \vec{A}(\vec{x}) = \frac{\mu_0 wa\sigma}{4\pi}\int\frac{\sin\theta'\,dA'}{|\vec{x}-\vec{x}'|}\hat{\phi}$$

Problem: $\hat{\phi}$ is a function, but in curvilinear coordinates there's no good way to write what it depends on.

Two solutions.

I) Rewrite $\hat{\phi}$ (or $\hat{r}, \hat{\theta}$) in terms of $\hat{\imath}, \hat{\jmath}, \hat{k}$

$$\hat{\phi} = \cos\theta\cos\phi\,\hat{\imath} + \cos\theta\sin\phi\,\hat{\jmath} - \sin\theta\,\hat{k}$$

Get three integrals: $A_x\hat{\imath} = \text{stuff} \int \text{stuff}\,\hat{\imath}$
$A_y\hat{\jmath} = \text{stuff} \int \text{stuff}\,\hat{\jmath}$ etc

Brute force method; can be easy if you know for physical reasons that some of the pieces have to be zero.

II) Be clever.

In this case, back things up. Instead of writing

$\vec{v} = wa\sin\theta'\hat{\phi}$, note that w points in the $\hat{k}$ direction and $\vec{x}'$ points in the $\hat{r}$ direction. $\hat{k} \times \hat{r} = \hat{\phi}$ (it's the only direction perpendicular to both)

It is also the case that the angle θ' is the angle between R (and thus $\vec{\omega}$) and $\vec{x}'$. And since in general $\vec{A} \times \vec{B} = AB\sin\theta\,(\widehat{\text{direction}})$

we can infer that $\vec{v} = \vec{\omega} \times \vec{x}'$ (SUPER NOT OBVIOUS), $\vec{K} = \sigma\vec{\omega} \times \vec{x}'$

Also note $dA' = a^2 d\Omega'$ and put it together:

$$\vec{A}(\vec{x}) = \frac{\mu_0 \sigma a^2}{4\pi}\,\vec{\omega} \times \int \frac{\vec{x}'\,d\Omega'}{|\vec{x} - \vec{x}'|}$$

So now our integrand is a little smaller. Define $\int \frac{\vec{x}'\,d\Omega'}{|\vec{x} - \vec{x}'|} \equiv \vec{f}(\vec{x})$

The integral has to return a vector result and can only depend on $\vec{x}$ ($\vec{x}'$ integrates out) (note $\vec{x}$ trapped in $|\vec{x} - \vec{x}'|$, but it's okay)

Even tricksier, I claim $\vec{f}(\vec{x}) = C\vec{x}$ (which is to say $\vec{f}(\vec{x})$ points in the $\vec{x}$ direction)

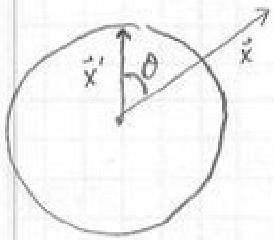
Why? We integrate over a sphere. $\vec{x}$ is the only preferred direction in the problem, and there's nothing like a cross product to shoot things off in another direction. Where else could it point? (basically a sophisticated symmetry argument)

More hax: If $\vec{f}(\vec{x}) = C\vec{x}$, consider $\vec{x} \cdot \vec{f} = \vec{x} \cdot C\vec{x} = Cr^2$,

so $$C = \frac{\vec{x} \cdot \vec{f}}{r^2} = \frac{1}{r^2} \int \frac{\vec{x} \cdot \vec{x}'\,d\Omega'}{|\vec{x} - \vec{x}'|}$$

So now the strategy reveals itself: If you can work a dot product into the game, you can invoke things like $\vec{A}\cdot\vec{B} = AB\cos\theta$

Reorient the sphere so that $\vec{x}'$ points along the z-axis
(you can do that for the purposes of this integral without affecting other stuff)



Now $\vec{x}\cdot\vec{x}' = ra\cos\theta$ and

$$C = \frac{a}{r}\int_0^{\pi} \frac{(\cos\theta)(2\pi)\sin\theta\, d\theta}{\sqrt{r^2+a^2-2ra\cos\theta}} \qquad (2\pi \text{ from } \phi \text{ integral})$$

Do the integral whichever way, get $C = \frac{4\pi}{3a}$ $\quad r<a$

$\frac{4\pi a^2}{3r^3}$ $\quad r>a$

So since $\vec{A}(\vec{x}) = \frac{\mu_0\sigma a^2}{4\pi}\,\vec{\omega}\times C\vec{x}$ we get

$$\boxed{\begin{aligned}\vec{A} &= \frac{\mu_0\sigma a}{3}\,\vec{\omega}\times\vec{x} \quad \text{for } r<a \\ &\frac{\mu_0\sigma a^4}{3r^3}\,\vec{\omega}\times\vec{x} \quad \text{for } r>a\end{aligned}}$$

Was that easier than brute forcing $\hat{\imath}, \hat{\jmath}, \hat{k}$? /shrug

<u>One last thing: Gauges</u>

V is arbitrary up to a scalar constant C because only ΔV and ∇V are physical. So V and $V+C$ are equivalent. We often impose an arbitrary but pleasant condition to fix C.

For example, impose $V\to 0$ as $r\to\infty$ to get $\frac{kq}{r}$ instead of $\frac{kq}{r}+5$

$\vec{A}$ is arbitrary up to even more. $\vec{B}$ is what's physical, and

$\vec{B} = (\vec{\nabla}\times\vec{A})$, so if we add something curl-free to $\vec{A}$, it changes nothing

∇f for any f is curl-free, so $\vec{A}$ is arbitrary up to some ∇f

$\vec{B} = \vec{\nabla}\times\vec{A} = \vec{\nabla}\times(\vec{A}+\nabla f)$

Conditions that we impose to remove ambiguity are called gauge conditions.
We speak of operating in a particular gauge.

In general, $\vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{x}')\, d^3x'}{|\vec{x} - \vec{x}'|} + \nabla f$

Now, $\vec{\nabla} \cdot \frac{\mu_0}{4\pi} \int \frac{J\, d^3x'}{|\vec{x}-\vec{x}'|} = 0$ (book proves it, follows from $\vec{\nabla} \cdot \vec{J} = 0$ in statics)

But $\nabla \cdot \nabla f \neq 0$ in general. So we could impose the Coulomb gauge condition: that $\vec{\nabla} \cdot \vec{A} = 0$

Why? Mostly so we can get this:

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} \Rightarrow \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \mu_0 \vec{J}$$

$$\Rightarrow \vec{\nabla}(\cancel{\vec{\nabla} \cdot \vec{A}}) - \vec{\nabla}^2 \vec{A} = \mu_0 \vec{J}$$

$$\Rightarrow \boxed{\nabla^2 \vec{A} = -\mu_0 \vec{J}}$$

In Cartesian, this means $\nabla^2 A_x = -\mu_0 J_x$ etc and is Poisson's eqn for $A + J$

In dynamics, when $\vec{\nabla} \cdot \vec{J} \neq 0$, we'll use more interesting gauges.