1. Problem 1

(a) Assume that, $u(x,t) = F(x)G(t)$ then $u_{xx} = F''(x)G(t)$ and $u_t = F(x)G'(t)$ and the 1-D heat equation becomes,

$$\frac{G'(t)}{c^2 G(t)} = \frac{F''(x)}{F(x)} = -\lambda, \tag{1}$$

where we have introduced the separation constant λ.[1] From this equation we have the two ODE's,

$$G'(t) + \lambda c^2 G(t) = 0, \tag{2}$$
$$F''(x) + \lambda F(x) = 0. \tag{3}$$

Each of these ODE's can be solved through 'elementary methods' to get,[2]

$$\lambda \in \mathbb{R} \;:\; G(t) = Ae^{-\lambda c^2 t}, \quad A \in \mathbb{R}, \tag{4}$$
$$\lambda \in \mathbb{R}^+ \;:\; F(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x), \tag{5}$$
$$\lambda \in \mathbb{R}^- \;:\; F(x) = c_3 \cosh(\sqrt{|\lambda|}x) + c_4 \sinh(\sqrt{|\lambda|}x), \tag{6}$$
$$\lambda = 0 \;:\; F(x) = c_5 + c_6 x. \tag{7}$$

Each of the functions $F(x)$ must also satisfy the boundary conditions, $u_x(0,t) = 0$ and $u_x(L,t)$ and so we won't need all of them. Notice that the boundary conditions imply that,

$$u_x(0,t) = F'(0)G(t) = 0, \tag{8}$$
$$u_x(L,t) = F'(L)G(t) = 0, \tag{9}$$

which gives $F'(0) = 0$ and $F'(L) = 0$.[3] So, we now have to determine, which of the previous functions, $F(x)$, satisfy these boundary conditions. To this end we have the following arguments,

$$\begin{aligned}
\lambda \in \mathbb{R}^+ \;&:\; F'(0) = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}0) + c_2\sqrt{\lambda}\cos(\sqrt{\lambda}0) = c_1\sqrt{\lambda}\cdot 0 + c_2\sqrt{\lambda}\cdot 1 \Rightarrow c_2 = 0,\\
\lambda \in \mathbb{R}^+ \;&:\; F'(L) = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}L) + c_2\sqrt{\lambda}\cos(\sqrt{\lambda}L) = c_1\sqrt{\lambda}\cdot\sin(\sqrt{\lambda}L) + 0\cdot\sqrt{\lambda}\cos(\sqrt{\lambda}L) \Rightarrow\\
&\Rightarrow\; c_1\sqrt{\lambda}\cdot\sin(\sqrt{\lambda}L) = 0 \iff c_1 = 0 \;\underline{\text{or}}\; \sin(\sqrt{\lambda}L) = 0.
\end{aligned}$$

If we consider the case that $c_1 = 0$ then we have $F(x) = 0$ for $\lambda \in \mathbb{R}^+$ but we should try to keep as many solutions as possible and we ignore this case. Thus assume that $c_1 \neq 0$ we have that $\sin(\sqrt{\lambda}L) = 0$, which is true for $\sqrt{\lambda} = n\pi/L$ and we have the following eigenvalue/eigenfunction pairs indexed by n,

$$F_n(x) = c_n \cos(\sqrt{\lambda}x), \quad \lambda_n = \frac{n^2\pi^2}{L^2}, \quad n = 1,2,3,\ldots. \tag{10}$$

We now consider the $\lambda \in \mathbb{R}^-$ case to find that,

$$\begin{aligned}
\lambda \in \mathbb{R}^- \;&:\; F'(x) = c_3\sqrt{|\lambda|}\sinh(\sqrt{|\lambda|}0) + c_4\sqrt{|\lambda|}\cosh(\sqrt{|\lambda|}0) = c_3\cdot 0 + c_4\cdot 1 = 0 \Rightarrow c_4 = 0,\\
\lambda \in \mathbb{R}^- \;&:\; F'(x) = c_3\sqrt{|\lambda|}\sinh(\sqrt{|\lambda|}L) + c_4\sqrt{|\lambda|}\cosh(\sqrt{|\lambda|}L) = c_3\sqrt{|\lambda|}\sinh(\sqrt{|\lambda|}L) + 0\cdot\sqrt{|\lambda|}\cosh(\sqrt{|\lambda|}L) =\\
&= c_3\sqrt{|\lambda|}\frac{e^{\sqrt{|\lambda|}L} - e^{-\sqrt{|\lambda|}L}}{2} = 0 \Rightarrow c_3 = 0,
\end{aligned}$$

which means that for $\lambda \in \mathbb{R}^-$ we only have the trivial solution $F(x) = 0$. Lastly, we consider the case $\lambda = 0$ to get,

$$\lambda = 0 \;:\; F'(0) = F'(L) = c_6 = 0 \Rightarrow c_5 \in \mathbb{R}, \tag{11}$$

[1] This occurs in conjunction with the following argument. Since (25) must be true for all (x,t) then both sides must be equal to a function that has neither t's nor x's. Hence they must be equal to a constant function. To see that this is true put an x or t on the side that has λ and test points.

[2] These elementary methods are those you learned in ODE's and can be found in the solutions to Homework 9 problem 1a.

[3] We assume that $G(t) = 0$ because if it did then we would have $u(x,t) = F(x)G(t) = F(x)\cdot 0 = 0$, which is called the trivial solution and is ignored since it is already in thermal equilibrium. We care about dynamics!

which gives the last eigenpair,[4]

$$F_0 = c_0 \in \mathbb{R} \quad \lambda_0 = 0. \tag{12}$$

Noting that there are infinitely many λ's implies now that there are infinitely many temporal solutions (4) and we have,

$$G_n(t) = A_n e^{\lambda_n c^2 t}, \quad \lambda = \frac{n^2\pi^2}{L^2}, \; n = 1, 2, 3, \ldots. \tag{13}$$

For the case where $\lambda = 0$ we have the ODE $G'(t) = 0$, whose solution is$G_0(t) = A_0 \in \mathbb{R}$. Thus we have infinitely many functions, that solve the PDE, of the form:

$$u_n(x,t) = F_n(x)G_n(t), \; n = 0, 1, 2, 3, \ldots \tag{14}$$

Hence since the PDE is linear superposition implies that we have the general solution,

$$\begin{aligned}
u(x,t) &= \sum_{n=0}^{\infty} u_n(x,t) = u_0(x,t) + \sum_{n=1}^{\infty} u_n(x,t) && (15)\\
&= F_0(x)G_0(x) + \sum_{n=1}^{\infty} F_n(x)G_n(t) && (16)\\
&= c_0 \cdot A_0 + \sum_{n=1}^{\infty} A_n c_n \cos(\sqrt{\lambda_n}x)e^{\lambda_n c^2 t} && (17)\\
&= a_0 + \sum_{n=1}^{\infty} a_0 \cos(\sqrt{\lambda_n}x)e^{\lambda_n c^2 t}, && (18)
\end{aligned}$$

which is the general solution of the heat equation with the given boundary conditions.

(b) Describe the long term behaviour as k is increased and as ρ is increased.

If k(thermal conductivity) is increased, the temporal solution decays faster and the system reaches equilibrium sooner.

If ρ(density) is increased, the temporal solution decays slower and the system takes longer to reach equilibrium.

(c) To find the unknown constants present in the general solution we must apply an initial condition, $u(x,0) = f(x)$. Doing so gives,

$$\begin{aligned}
u(x,0) = f(x) &= a_0 + \sum_{n=1}^{\infty} a_0 \sin(\sqrt{\lambda_n}x)e^{\lambda_n c^2 \cdot 0} && (19)\\
&= a_0 + \sum_{n=1}^{\infty} a_0 \cos(\sqrt{\lambda_n}x), && (20)
\end{aligned}$$

which is a Fourier cosine half-range expansion of the initial condition. Thus the unknown constants are Fourier coefficients and,

$$\begin{aligned}
a_0 &= \frac{1}{L}\int_0^L f(x)dx, && (21)\\
a_n &= \frac{2}{L}\int_0^L f(x)\cos(\sqrt{\lambda_n}x)dx. && (22)
\end{aligned}$$

If we note homework7 problem1 then we known these Fourier coefficients as,

$$\begin{aligned}
a_0 &= \frac{k}{2}, && (23)\\
a_n &= \frac{4k}{n^2\pi^2}\left[2\cos\left(\frac{n\pi}{2}x\right) - (-1)^n - 1\right]. && (24)
\end{aligned}$$

Moreover, if we take $L = k = 1$ we see that $\lim_{t\to\infty} u(x,t) = a_0 = .5$, which implies that under these insulating boundary conditions the equilibrium state for the medium is a constant function $u = .5$ and that this is nothing more than the average of the initial configuration.

[4] Here we have used the subscripts to denote that these are all associated with the $\lambda = 0$ case. We have also trivially changed c_5 to c_0.

2. Problem 2 : Recall the 1-D conservation law

$$\frac{\partial u}{\partial t} = -k\frac{\partial \phi}{\partial x} \qquad (8)$$

(a) Assume that ϕ is proportional to u, to derive the convection/transport equation $u_t + cu_x = 0$

$$\begin{aligned}
\phi &= \alpha u \\
\frac{\partial \phi}{\partial x} &= \alpha\frac{\partial u}{\partial x} \\
\frac{\partial u}{\partial t} &= -k\frac{\partial Q}{\partial x} \Rightarrow \frac{\partial u}{\partial t} = -\alpha k\frac{\partial u}{\partial x} \Rightarrow u_t + cu_x = 0
\end{aligned}$$

(b) Given that $u(x,0) = u_0(x)$ show that $u(x,t) = u_0(x-ct)$ is a solution.

$$\begin{aligned}
u(x,t) &= u_0(x-ct) \\
u_t &= -cu_0' \qquad u_x = u_0' \\
u_t + cu_x &= -cu_0' + cu_0' = 0
\end{aligned}$$

(c) If $\phi(x,t) = cu - du_x$, derive from (8) the convection-diffusion equation $u_t + cu_x - du_{xx}$

$$\begin{aligned}
\frac{\partial \phi}{\partial x} &= c\frac{\partial u}{\partial x} - d\frac{\partial^2 u}{\partial x^2} \\
\frac{\partial u}{\partial t} &= -k\frac{\partial \phi}{\partial x} \Rightarrow \frac{\partial u}{\partial t} = -c\frac{\partial u}{\partial x} + d\frac{\partial^2 u}{\partial x^2} \\
&\Rightarrow u_t + cu_x - du_{xx} = 0
\end{aligned}$$

(d)

$$u_t = Du_{xx} - cu_x - \lambda u \qquad (9)$$

Assume that $u(x,t) = w(x,t)e^{\alpha x - \beta t}$ and show that (a) can be transformed into a heat equation on the variable w where $\alpha = \frac{c}{2D}$ and $\beta = \lambda + \frac{c^2}{4D}$

$$\begin{aligned}
u_t &= w_t e^{\alpha x - \beta t} + w\beta e^{\alpha x - \beta t} \\
u_x &= w_x e^{\alpha x - \beta t} + w\alpha e^{\alpha x - \beta t} \\
u_{xx} &= w_{xx} e^{\alpha x - \beta t} + 2w_x\alpha e^{\alpha x - \beta t} + w\alpha^2 e^{\alpha x - \beta t} \\
u_t &= Du_{xx} - cu_x - \lambda u \\
w_t e^{\alpha x - \beta t} - w\beta e^{\alpha x - \beta t} &= Dw_{xx}e^{\alpha x - \beta t} + D2w_x\alpha e^{\alpha x - \beta t} + Dw\alpha^2 e^{\alpha x - \beta t} - \\
&\quad -cw_x e^{\alpha x - \beta t} - cw\alpha e^{\alpha x - \beta t} - \lambda w e^{\alpha x - \beta t} \\
&\Rightarrow w_t - \beta w = Dw_{xx} + 2D\alpha w_x + Dw\alpha^2 - cw_x - c\alpha w - \lambda w \\
w_t &= Dw_{xx} + (2D\alpha - c)w_x + (\beta - c\alpha + D\alpha^2 - \lambda)w \\
w_t &= Dw_{xx} + 2D\left(\frac{c}{2D} - c\right)w_x + \left(\lambda + \frac{c^2}{4D} - \frac{c^2}{2D} + \frac{Dc^2}{4D^2}\right)w \\
w_t &= Dw_{xx} \leftarrow \text{ heat equation on variable } w
\end{aligned}$$

3. Problem 3 : The only difference between this problem and problem 1 are the time dynamics specified by the PDE. This gives a second-order ODE in time and from this ODE we have oscillations of Fourier modes instead of exponential decay. Work showing this follows.

(a) Assume that, $u(x,t) = F(x)G(t)$ then $u_{xx} = F''(x)G(t)$ and $u_t = F(x)G''(t)$ and the 1-D heat equation becomes,

$$\frac{G''(t)}{c^2 G(t)} = \frac{F''(x)}{F(x)} = -\lambda, \qquad (25)$$

where we have introduced the separation constant λ.[5] From this equation we have the two ODE's,

$$G''(t) + \lambda c^2 G(t) = 0, \tag{26}$$
$$F''(x) + \lambda F(x) = 0. \tag{27}$$

Each of these ODE's can be solved through 'elementary methods' to get,[6]

$$\begin{aligned}
\lambda \in \mathbb{R}^+ &: \quad G(t) = A\cos(\sqrt{\lambda}ct) + A^* \sin(\sqrt{\lambda}ct), \quad A, A^* \in \mathbb{R}, && (28)\\
\lambda = 0 &: \quad G(t) = A + A^* t, A, A^* \in \mathbb{R}, && (29)\\
\lambda \in \mathbb{R}^+ &: \quad F(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x), && (30)\\
\lambda \in \mathbb{R}^- &: \quad F(x) = c_3 \cosh(\sqrt{|\lambda|}x) + c_4 \sinh(\sqrt{|\lambda|}x), && (31)\\
\lambda = 0 &: \quad F(x) = c_5 + c_6 x. && (32)
\end{aligned}$$

Notice there are different time functions than before. This highlights the departure from the heat equation dynamics.[7] Each of the functions $F(x)$ must also satisfy the boundary conditions, $u_x(0,t) = 0$ and $u_x(L,t)$ and so we won't need all of them. Notice that the boundary conditions imply that,

$$u_x(0,t) = F'(0)G(t) = 0, \tag{33}$$
$$u_x(L,t) = F'(L)G(t) = 0, \tag{34}$$

which gives $F'(0) = 0$ and $F'(L) = 0$.[8] So, we now have to determine, which of the previous functions, $F(x)$, satisfy these boundary conditions. To this end we have the following arguments,

$$\begin{aligned}
\lambda \in \mathbb{R}^+ &: \quad F'(0) = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}0) + c_2\sqrt{\lambda}\cos(\sqrt{\lambda}0) = c_1\sqrt{\lambda}\cdot 0 + c_2\sqrt{\lambda}\cdot 1 \Rightarrow c_2 = 0,\\
\lambda \in \mathbb{R}^+ &: \quad F'(L) = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}L) + c_2\sqrt{\lambda}\cos(\sqrt{\lambda}L) = c_1\sqrt{\lambda}\cdot\sin(\sqrt{\lambda}L) + 0\cdot\sqrt{\lambda}\cos(\sqrt{\lambda}L) \Rightarrow\\
&\Rightarrow \quad c_1\sqrt{\lambda}\cdot\sin(\sqrt{\lambda}L) = 0 \iff c_1 = 0 \ \underline{\text{or}}\ \sin(\sqrt{\lambda}L) = 0.
\end{aligned}$$

If we consider the case that $c_1 = 0$ then we have $F(x) = 0$ for $\lambda \in \mathbb{R}^+$ but we should try to keep as many solutions as possible and we ignore this case. Thus assume that $c_1 \neq 0$ we have that $\sin(\sqrt{\lambda}L) = 0$, which is true for $\sqrt{\lambda} = n\pi/L$ and we have the following eigenvalue/eigenfunction pairs indexed by n,

$$F_n(x) = c_n \cos(\sqrt{\lambda}x), \quad \lambda_n = \frac{n^2\pi^2}{L^2}, \quad n = 1, 2, 3, \ldots. \tag{35}$$

We now consider the $\lambda \in \mathbb{R}^-$ case to find that,

$$\begin{aligned}
\lambda \in \mathbb{R}^- &: \quad F'(x) = c_3\sqrt{|\lambda|}\sinh(\sqrt{|\lambda|}0) + c_4\sqrt{|\lambda|}\cosh(\sqrt{|\lambda|}0) = c_3\cdot 0 + c_4\cdot 1 = 0 \Rightarrow c_4 = 0,\\
\lambda \in \mathbb{R}^- &: \quad F'(x) = c_3\sqrt{|\lambda|}\sinh(\sqrt{|\lambda|}L) + c_4\sqrt{|\lambda|}\cosh(\sqrt{|\lambda|}L) = c_3\sqrt{|\lambda|}\sinh(\sqrt{|\lambda|}L) + 0\cdot\sqrt{|\lambda|}\cosh(\sqrt{|\lambda|}L) =\\
&= \quad c_3\sqrt{|\lambda|}\frac{e^{\sqrt{|\lambda|}L} - e^{-\sqrt{|\lambda|}L}}{2} = 0 \Rightarrow c_3 = 0,
\end{aligned}$$

which means that for $\lambda \in \mathbb{R}^-$ we only have the trivial solution $F(x) = 0$. Lastly, we consider the case $\lambda = 0$ to get,

$$\lambda = 0 \quad : \quad F'(0) = F'(L) = c_6 = 0 \Rightarrow c_5 \in \mathbb{R}, \tag{36}$$

which gives the last eigenpair,[9]

$$F_0 = c_0 \in \mathbb{R} \quad \lambda_0 = 0. \tag{37}$$

[5]This occurs in conjunction with the following argument. Since (25) must be true for all (x,t) then both sides must be equal to a function that has neither t's nor x's. Hence they must be equal to a constant function. To see that this is true put an x or t on the side that has λ and test points.

[6]These elementary methods are those you learned in ODE's and can be found in the solutions to Homework 9 problem 1a.

[7]Based on problem1 and our studies in class we do not seek to find a time function for $\lambda < 0$ since we know that the spatial function associated with these eigenvalues will not satisfy the boundary conditions.

[8]We assume that $G(t) = 0$ because if it did then we would have $u(x,t) = F(x)G(t) = F(x)\cdot 0 = 0$, which is called the trivial solution and is ignored since it is already in thermal equilibrium. We care about dynamics!

[9]Here we have used the subscripts to denote that these are all associated with the $\lambda = 0$ case. We have also trivially changed c_5 to c_0.

Noting that there are infinitely many λ's implies now that there are infinitely many temporal solutions (28) and we have,

$$G_n(t) = A_n \cos(\sqrt{\lambda_n} ct) + A_n^* \sin(\sqrt{\lambda_n} ct), \quad \lambda = \frac{n^2\pi^2}{L^2}, \ n = 1, 2, 3, \ldots. \tag{38}$$

For the case where $\lambda = 0$ we have the ODE $G''(t) = 0$, whose solution is$G_0(t) = A_0 + A_0^* t$. Thus we have infinitely many functions, that solve the PDE, of the form:

$$u_n(x,t) = F_n(x)G_n(t), \ n = 0, 1, 2, 3, \ldots \tag{39}$$

Hence since the PDE is linear superposition implies that we have the general solution,

$$\begin{aligned} u(x,t) &= \sum_{n=0}^{\infty} u_n(x,t) = u_0(x,t) + \sum_{n=1}^{\infty} u_n(x,t) && (40) \\ &= F_0(x)G_0(x) + \sum_{n=1}^{\infty} F_n(x)G_n(t) && (41) \\ &= c_0 \cdot (A_0 + A_0^* t) + \sum_{n=1}^{\infty} A_n c_n \cos(\sqrt{\lambda_n} x) e^{\lambda_n c^2 t} && (42) \\ &= a_0 + a_0^* t + \sum_{n=1}^{\infty} \left[a_n \cos(\sqrt{\lambda_n} ct) + a_n^* \sin(\sqrt{\lambda_n} ct) \right] \cos(\sqrt{\lambda_n} x), && (43) \end{aligned}$$

which is the general solution of the wave equation with the given boundary conditions.

(b) To find the unknown constants present in the general solution we must apply an initial condition, $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$. Noting that $g(x) = 0$, since there is no initial velocity, implies that $a_0^* = a_n^* = -$. Calculating the rest we find,

$$\begin{aligned} u(x,0) = f(x) &= a_0 + \sum_{n=1}^{\infty} a_0 \sin(\sqrt{\lambda_n} x) e^{\lambda_n c^2 \cdot 0} && (44) \\ &= a_0 + \sum_{n=1}^{\infty} a_0 \cos(\sqrt{\lambda_n} x), && (45) \end{aligned}$$

which is a Fourier cosine half-range expansion of the initial condition. Thus the unknown constants are Fourier coefficients and,

$$\begin{aligned} a_0 &= \frac{1}{L} \int_0^L f(x) dx, && (46) \\ a_n &= \frac{2}{L} \int_0^L f(x) \cos(\sqrt{\lambda_n} x) dx. && (47) \end{aligned}$$

If we note homework7 problem1 then we known these Fourier coefficients as,

$$\begin{aligned} a_0 &= \frac{k}{2}, && (48) \\ a_n &= \frac{4k}{n^2\pi^2} \left[2\cos\left(\frac{n\pi}{2} x \right) - (-1)^n - 1 \right]. && (49) \end{aligned}$$

4. We begin with,

$$dspu(x,t) = \frac{1}{2} \left[u_0(x - ct) + u_0(x + ct) \right] + \frac{1}{2c} \int_{x-ct}^{x+ct} v_0(y) dy. \tag{50}$$

To show that this is a solution to the wave equation we find its partial derivatives, u_{tt} and u_{xx} and see if they maintain equality in the wave equation. First note that the chain rule implies,

$$\begin{aligned} \frac{\partial}{\partial t} u_0(x - ct) &= \frac{\partial u_0(x - ct)}{\partial [x - ct]} \frac{\partial [x - ct]}{\partial t} && (51) \\ &= u_0'(x - ct) \cdot (-c), && (52) \end{aligned}$$

and that the chain-rule with the fundamental theorem of calculus implies,

$$\frac{\partial}{\partial t}\int_0^{x+ct} v_0(y)dy = v_0(x+ct)\cdot\frac{\partial[x+ct]}{\partial t} \quad (53)$$
$$= v_0(x+ct)\cdot c. \quad (54)$$

Thus we have that,

$$u_{tt} = \frac{1}{2}\left[c^2u_0''(x-ct)+c^2u_0''(x+ct)\right]+\frac{1}{2c}\left[c^2v_0'(x+ct)-c^2v_0'(x-ct)\right], \quad (55)$$
$$u_{xx} = \frac{1}{2}\left[u_0''(x-ct)+u_0''(x+ct)\right]+\frac{1}{2c}\left[v_0'(x+ct)-v_0'(x-ct)\right], \quad (56)$$

which implies that $u_{tt} = c^2u_{xx}$.

5. Consider the non-homogeneous 1-D wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}+F(x,t) \qquad (10)$$

Letting F(x,t)=Asin(wt) gives the following Fourier Series for F

$$F(s,t) = \sum_{n=-1}^{\infty} f_n(t)sin\left(\frac{n\pi x}{L}\right) \qquad (14)$$
$$f_n(t) = \frac{2A}{n\pi}(1-(-1)^n)sin(wt) \qquad (15)$$

(a) Show that substituting (14)-(15) into (10) gives

$$G_n+\left(\frac{cn\pi}{L}\right)^2G_n = \frac{2A}{n\pi}(1-(-1)^n)sin(wt) \qquad (16)$$

$$F(x,t) = F_n(x)f_n(t) = \sum_{n=1}^{\infty} f_n(t)sin\left(\frac{n\pi x}{L}\right)$$
$$\Rightarrow F_n(x) = \sum_{n=1}^{\infty} sin\left(\frac{n\pi x}{L}\right)$$
$$u(x,t) = F_n(t)G_n(t)$$
$$\frac{\partial^2 u}{\partial t^2} = F_n(t)G_n''(t) = \sum_{n=1}^{\infty} G_n''(t)sin\left(\frac{n\pi x}{L}\right)$$
$$\frac{\partial^2 u}{\partial x^2} = F_n''(t)G_n(t) = \sum_{n=1}^{\infty} -\left(\frac{L}{n\pi}\right)^2G_n(t)sin\left(\frac{n\pi x}{L}\right)$$
$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}+F(x,t)$$
$$\Rightarrow \sum_{n=1}^{\infty} G_n''sin\left(\frac{n\pi x}{L}\right) = \sum_{n=1}^{\infty} G_n\left(\frac{cn\pi}{L}\right)^2 sin\left(\frac{n\pi}{L}\right)+\sum_{n=1}^{\infty} f_n(t)sin\left(\frac{n\pi x}{L}\right)$$
$$\Rightarrow \sum_{n=1}^{\infty} G_n''sin\left(\frac{n\pi x}{L}\right) = \sum_{n=1}^{\infty}\left[-\left(\frac{cn\pi}{L}\right)^2G_n+f_n(t)\right]sin\left(\frac{n\pi x}{L}\right)$$

↑ for this statement to be true, the coefficients must be equal

$$\Rightarrow G_n'' = -\left(\frac{cn\pi}{L}\right)^2G_n+f_n(t)$$
$$\Rightarrow G_n''+\left(\frac{cn\pi}{L}\right)^2G_n = \frac{2A}{n\pi}(1-(-1)^n)sin(wt)$$

(b) The solution to (16) is given by

$$G(t) = B_ncos\left(\frac{cn\pi}{L}t\right)+B_n^\star sin\left(\frac{cn\pi}{L}t\right)+G_p(t)$$

i. If $w \neq \frac{cn\pi}{L}$, what would be your choice for $G_p(t)$if you were using the method of undetermined coefficients?

$$G_p(t) = Acos(wt) + Bsin(wt)$$

ii. If $w = \frac{cn\pi}{L}$ what would be your choice for $G_p(t)$?

$$G_p(t) = Atcos\left(\frac{cn\pi}{L}t\right) + Btsin\left(\frac{cn\pi}{L}\right)$$

iii. For (ii), what is the $\lim_{t\to\infty} u(x,t)$?

$$\lim_{t\to\infty} u(x,t) = \infty$$

iv. What does this limit imply physically?

This is called resonance and implies that the magnitude of oscillation approaches infinity as t gets larger and likely the object/string would break under these forces.