Origin of group velocity

we know that phase velocity is $v_{ph} = \omega/k$

group velocity is $v_g = d\omega/dk$.

expect that the center of a pulse envelope travels at v_g. $\rightarrow E_0(t) \rightarrow E_0(t - z/v_g)$

Note that t is relative to the pulse peak:

$e^{-t^2/\tau^2} \rightarrow e^{-(t-z/v_g)^2/\tau^2}$ moving peak.

Derive group velocity:

after propagation by L

$$E_{out}(\omega) = \underbrace{E_0 \tau \sqrt{\pi}\, e^{-\tau^2(\omega-\omega_0)^2/4}}_{\text{input pulse spectrum}} \underbrace{e^{i\omega n(\omega)L/c}}_{\text{propagation effect}}$$

To describe pulse in the time domain, we must transform back.

- But $e^{i\omega n/c}$ is complicated

$\therefore$ approximate

Define spectral phase: $\phi(\omega) = \omega n(\omega) L/c$ L is fixed

$\phi(\omega)$ varies slowly away from resonance

Taylor-expand around ω_0

$$\phi(\omega) \approx \phi(\omega_0) + (\omega-\omega_0)\phi'(\omega)\Big|_{\omega_0} + \frac{1}{2!}(\omega-\omega_0)^2 \phi''(\omega)\Big|_{\omega_0} + O(\Delta\omega^3)$$

$\phi(\omega_0) = \omega_0 n(\omega_0) L/c = k_0 n(\omega_0) L$ = constant phase shift, eval @ ω_0

$\phi'(\omega_0)$ = units of time

= group delay

$\phi''(\omega_0)$ = group delay dispersion.

Keep 1st order, transform back to time domain

$$\tilde{E}_{out}(\omega) = \tilde{E}_{in}(\omega)\, e^{ik_0 n_0 L}\, e^{i(\omega-\omega_0)\phi'(\omega_0)}$$

$$E_{out}(t) = e^{ik_0 n_0 L}\, \mathfrak{F}^{-1}\left\{ \tilde{E}_{in}(\omega)\, e^{i(\omega-\omega_0)\phi'(\omega_0)} \right\}$$

shift theorem:

$$E_{out}(t) = e^{ik_0 n_0 L}\, E_0 \tau\sqrt{\pi}\, \mathfrak{F}^{-1}\left\{ e^{-(\omega-\omega_0)^2\tau^2/4}\, e^{i(\omega-\omega_0)\phi'(\omega_0)} \right\}$$

$$\text{from } \mathfrak{F}^{-1}\{ F(\omega-\omega_0)\} = f(t)\, e^{-i\omega_0 t}$$

$$E_{out}(t) = E_0 \tau\sqrt{\pi}\, e^{i(k_0 n_0 L - \omega_0 t)}\, \mathfrak{F}^{-1}\left\{ e^{-\omega'^2\tau^2/4}\, e^{i\omega'\phi'(\omega_0)} \right\}$$

we know that

$$\mathfrak{F}\left\{ e^{-t^2/\tau^2} \right\} = \tau\sqrt{\pi}\, e^{-\omega^2\tau^2/4}$$

$$\therefore \mathfrak{F}^{-1}\left\{ e^{-\omega^2\tau^2/4} \right\} = \frac{1}{\tau\sqrt{\pi}}\, e^{-t^2/\tau^2}$$

also, we can use shift thm in inverse:

$$\mathfrak{F}^{-1}\left\{ F(\omega)\, e^{i\omega t_0} \right\} = f(t-t_0)$$

here $t_0 = \phi'(\omega_0)$ (group delay)

$$E_{out}(t) = E_0\, e^{i(k_0 n_0 L - \omega_0 t)}\, e^{-(t-t_0)^2/\tau^2}$$

$$t_0 = \frac{d}{d\omega}(k_0 n(\omega))\, L = L/v_g$$

w/ $v_g \equiv d\omega/dk$

output pulse is:

$$E_{out}(t) = E_{in}(t - L/v_g)\, e^{i(k_0 n_0 L)}$$

our coordinate system moves at c
so that w/o dispersion, $E_{out}(t) = E_{in}(t)\, e^{i k_0 n_0 L}$

group delay $\tau_g(\omega) = \phi'(\omega)$ arrival time of a group near ω

note that any information must be communicated with some modulation
→ concept that $v_g < c$
even though $v_{ph} = \omega/k$ can be $> c$

Since τ_g varies w/ ω we will see some broadening of the pulse in addition to shift
- must keep next term in expansion of $\phi(\omega)$ to account

Pulse spreading ("chirp") (hw)

- Expand $\phi(\omega)$ to next order:
- next term is $\propto \frac{1}{2}(\omega-\omega_0)^2\,\phi''(\omega_0)$
- now $\mathfrak{F}^{-1}\left\{ e^{-\Delta\omega^2 \frac{\tau^2}{4}(1-i\Gamma)} \right\}$ use scale thm

result:

> pulse duration is $f(z)$
$$\tau(z) = \tau_0\sqrt{1+\Gamma^2}$$

> intensity $\propto 1/\tau(z)$

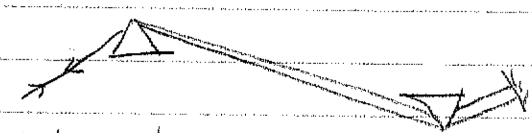
> new phase term: $\phi(t) = -bt^2$ $\quad b \sim \Gamma(z)/\tau^2(z)$ = chirp rate.

total phase: $k_0 n_0 z - \omega_0 t - bt^2$

$-\frac{\partial\phi}{\partial t}$ = instantaneous freq. $= \omega_0 + 2bt$

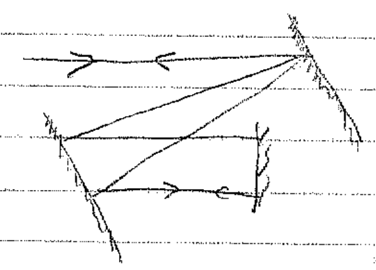
Problem: most materials stretch red first, blue last.
- need a way to compress

1) prisms



blue → shorter distance than red

2) gratings



Linear systems: transfer functions + impulse response.

consider propagation through a dispersive, absorbing medium:

$$\xrightarrow{E_{in}} \boxed{H(\omega)} \xrightarrow{E_{out}}$$

we have seen we get the output by:

$$E_{out}(\omega) = E_{in}(\omega)\, e^{i\frac{\omega}{c}\tilde{n}(\omega)L}$$

where $\tilde{n}(\omega)$ can be complex $\rightarrow$ absorption or gain

Let $H(\omega) \equiv e^{i\frac{\omega}{c}\tilde{n}(\omega)L}$ = transfer function

Time domain?

$$E_{out}(t) = \frac{1}{2\pi}\int E_{in}(\omega) H(\omega) e^{-i\omega t}\, d\omega$$

Convolution theorem:

$$\mathcal{F}^{-1}\{F(\omega)G(\omega)\} = \frac{1}{2\pi}\int d\omega\, e^{-i\omega t}\int dt'\, e^{i\omega t'} f(t')\int dt''\, e^{i\omega t''} g(t'')$$

$$= \int dt' f(t')\int dt'' g(t'')\int \frac{d\omega}{2\pi} e^{i\omega(t'+t'')} e^{-i\omega t}$$

$$= \int dt' f(t')\int dt'' g(t'')\, \underbrace{\delta(t-(t'+t''))}_{=\delta(t''-(t-t'))}$$

$$= \int dt' f(t')\, g(t-t')$$

this is a convolution. Symbolically: $f \otimes g$

Applied to our transfer function:

$$E_{out}(t) = \int E_{in}(t')\, h(t-t')\, dt'$$

where $h(t) = \mathcal{F}^{-1}\{H(\omega)\} \equiv$ impulse response.

suppose we put an impulse into system:

$E_{in}(t) \approx E_0\, \delta(t)$

$E_{in}(\omega) = E_0$ i.e. constant, all frequencies

then

$$E_{out}(t) = E_0 \int \delta(t')\, h(t-t')\, dt' = E_0\, h(t)$$

so $h(t)$ is the impulse response; connected to freq. response.