3_21_08

Note Title 3/20/2008

Problem : consider a particle of mass m in a bound state of the potential
$V(x) = -\alpha\,\delta(x)$.
What is the probability that a measurement of the momentum would yield a value $\geq p_0 = m\alpha/\hbar$?

we already showed that

$$\Psi(x,t) = \frac{\sqrt{m\alpha}}{\hbar}\, e^{-m\alpha|x|/\hbar^2}\, e^{-iEt/\hbar}$$

where $E = -m\alpha^2/2\hbar^2$.

Questions about p are easiest to handle in momentum space, so we transform $\Psi(x,t) \Rightarrow \Phi(p,t)$

$$\Phi(p,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \Psi(x,t)\, e^{-ipx/\hbar}\, dx$$

$$= \frac{\sqrt{m\alpha}}{\hbar}\, \frac{1}{\sqrt{2\pi\hbar}}\, e^{-iEt/\hbar} \int_{-\infty}^{\infty} e^{-m\alpha|x|/\hbar^2}\, e^{-ipx/\hbar}\, dx$$

$$\int_{-\infty}^{0} e^{(\underset{\uparrow\, p_0/\hbar}{m\alpha/\hbar^2} - ip/\hbar)x}\, dx + \int_{0}^{\infty} e^{-(\underset{\uparrow\, p_0/\hbar}{m\alpha/\hbar^2} + ip/\hbar)x}\, dx$$

$$\int_{-\infty}^{0} e^{(p_0 - ip)x/\hbar}\, dx + \int_{0}^{\infty} e^{-(p_0+ip)x/\hbar}\, dx$$

$$\frac{\hbar}{p_0 - ip} + \frac{\hbar}{p_0 + ip} = \frac{-\hbar(p_0+ip) - \hbar(p_0 - ip)}{p_0^2 + p^2}$$

$$= \frac{2\hbar p_0}{p_0^2 + p^2}$$

$$\Phi(p,t) = \frac{\sqrt{m\alpha}}{\cancel{\hbar}} \frac{1}{\sqrt{2\pi\hbar}} e^{-iEt/\hbar} \left(\frac{2\cancel{\hbar} p_0}{p_0^2 + p^2} \right)$$

$$= \sqrt{\frac{p_0}{2\pi}}\, e^{-iEt/\hbar} \frac{2p_0}{p_0^2+p^2}$$

$$\Phi(p,t) = \sqrt{\frac{2}{\pi}}\, e^{-iEt/\hbar}\, p_0^{3/2} \frac{1}{p_0^2 + p^2}$$

So, the probability we're looking for is

$$P[p \geq p_0] = \int_{p_0}^{\infty} |\Phi(p,t)|^2\, dp$$

$$\frac{2}{\pi} p_0^3 \int_{p_0}^{\infty} \frac{1}{(p_0^2 + p^2)^2}\, dp$$

mathematica to the rescue

$$\int \frac{1}{(p_0^2+p^2)^2}\,dp = \frac{1}{2p_0^3}\left[\frac{p\,p_0}{p_0^2+p^2} + \mathrm{Tan}^{-1}\left(\frac{p}{p_0}\right)\right]$$

$$\mathrm{Tan}^{-1}(1) = \pi/4 \qquad \tan^{-1}(\infty) = \frac{\pi}{2}$$

$$P[p \geq p_0] = \frac{2}{\pi}p_0^3 \cdot \frac{1}{2p_0^3}\Big[\qquad\qquad\Big]\Big|_{p_0}^{\infty}$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2} - \left(\frac{1}{2} + \frac{\pi}{4}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{4} - \frac{1}{2}\right] \approx \frac{1}{12} \approx .09$$

$$\psi(x, t=0) = \frac{A}{x^2+a^2}$$

$$\int_{-\infty}^{\infty} |\psi|^2\,dx = 1 = \frac{A^2\,\pi}{2\,a^3}$$

$$\Rightarrow \quad A^2 = \frac{2}{\pi}a^3$$

$$A = \sqrt{\frac{2}{\pi}}\,a^{3/2}$$

$\langle x \rangle = 0$

$$\langle x^2 \rangle = \frac{A^2 \pi}{2a} = \frac{2}{\pi} a^3 \cdot \frac{\pi}{2a} = a^2$$

$$\Rightarrow \quad \sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = a$$

$$\Phi(p,0) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \psi(x,0)\, e^{-ipx}\, dx$$

$$= \frac{A}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \frac{e^{-ipx/\hbar}}{x^2 + a^2}$$

$$= \frac{2A}{\sqrt{2\pi\hbar}} \int_0^{\infty} \frac{\cos(px/\hbar)}{x^2+a^2}\, dx$$

$$= \sqrt{\frac{a}{\hbar}}\, e^{-|p|a/\hbar}$$ already normalized

$\langle p \rangle = 0$

$$\langle p^2 \rangle = 2\frac{a}{\hbar} \int_0^{\infty} p^2 e^{-2pa/\hbar}\, dp$$

$$= \frac{2a}{\hbar}\, 2\left(\frac{\hbar}{2a}\right)^3 = \frac{\hbar^2}{2a^2}$$

$$\sigma_x \sigma_p = \sqrt{2}\,\frac{\hbar}{2} > \frac{\hbar}{2}$$