Radiation theory + retarded potentials

have discussed <u>propagation</u> of EM waves
now describe <u>generation</u> of waves.

Essential concept:

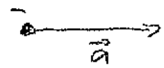
radiation comes from the <u>acceleration</u> of charges
- none from static or uniform motion.

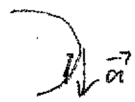
Examples:

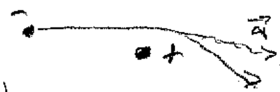
dipole radiation 

Larmor radiation formula: $P = \frac{2}{3} \frac{e^2 a^2}{c^3}$

antennas: linear, loop.

linear accelerators 

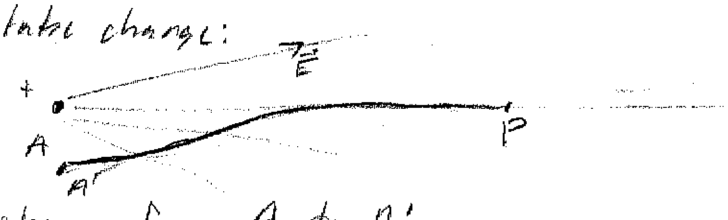
synchrotrons 

bremsstrahlung 

(braking radiation)

Retarded potentials and fields.

consider a static charge:



now move charge from A to A'
observation at P of field will see new field only
after disturbance propagates out at c
→ radiation.

Physical picture: field lines attached to charge
oscillation "shakes" the lines

Equations for scalar + vector potentials (HM 4.5)

$\vec{E} = -\nabla\phi$ static only

$\boxed{\vec{B} \equiv \nabla \times \vec{A}}$ always true

from $\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$

$\rightarrow \nabla \times \left(\vec{E} + \frac{1}{c}\frac{\partial \vec{A}}{\partial t}\right) = 0$

when $\nabla \times \vec{G} = 0$ $\vec{G}$ = arbitrary vector field

then we can write $\vec{G} = \nabla f$ f = potential.

$\therefore \quad -\nabla\Phi = \vec{E} + \frac{1}{c}\partial_t \vec{A}$

and $\boxed{\vec{E} = -\nabla\Phi - \frac{1}{c}\partial_t\vec{A}}$

Notice $\underline{\vec{A}' = \vec{A} + \nabla\xi}$ will give the same $\vec{B}$

choosing ξ is a choice of gauge

to get same $\vec{E}$:

$$\vec{E}' = -\nabla\Phi' - \frac{1}{c}\partial_t \overbrace{(\vec{A} + \nabla\xi)}^{\vec{A}'}$$

$$= -\nabla\underbrace{\left(\Phi' + \frac{1}{c}\partial_t\xi\right)}_{\Phi} - \frac{1}{c}\partial_t\vec{A} \quad = \vec{E}$$

$$\therefore \quad \underline{\Phi' = \Phi - \frac{1}{c}\partial_t\xi}$$

"gauge transformation": choose ξ $\vec{A} \rightarrow \vec{A}'$ $\Phi \rightarrow \Phi'$

We can choose another restriction on $\vec{A}$ w/o changing Maxwell:

Lorentz gauge:

$$\nabla\cdot\vec{A} + \frac{1}{c}\partial_t\Phi = 0$$

perform gauge transformation:

$$\nabla\cdot\vec{A}' + \frac{1}{c}\partial_t\Phi' = \nabla\cdot\vec{A} + \underline{\nabla^2\xi} + \frac{1}{c}\partial_t\Phi - \underline{\frac{1}{c^2}\partial_t^2\xi} = 0$$

$\therefore\ \nabla^2\xi - \frac{1}{c^2}\partial_t^2\xi = 0$ is a consequence of choice of Lorentz gauge.

Now we can calculate Φ and $\vec{A}$ from sources ρ and $\vec{J}$:
(in vacuum)

$$\nabla\cdot\vec{E} = 4\pi\rho \quad\rightarrow\quad \nabla\cdot\left(-\nabla\Phi - \frac{1}{c}\partial_t\vec{A}\right) = 4\pi\rho$$

since $\nabla\vec{A} = -\frac{1}{c}\partial_t\Phi$ in Lorentz gauge,

$$\rightarrow \boxed{\nabla^2\Phi - \frac{1}{c^2}\partial_t^2\Phi = -4\pi\rho}$$

similarly (see p142)

$$\underline{\nabla^2\vec{A} - \frac{1}{c^2}\partial_t^2\vec{A} = -\frac{4\pi}{c}\vec{J}}$$

With no local sources, Φ and $\vec{A}$ propagate according to wave equation at speed c.

In steady state, $\partial_t \to 0$, and

$$\left.\begin{aligned}\nabla^2\Phi &= -4\pi\rho\\ \nabla^2\vec{A} &= -\frac{4\pi}{c}\vec{J}\end{aligned}\right\}\ \text{Poisson eqn.}$$

static solutions:

$$\Phi(\vec{r}) = \int_V \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\,d^3r'$$

$$\vec{A}(\vec{r}) = \int_V J(\vec{r}\,')/|\vec{r}-\vec{r}\,'|\,d^3r'$$

sum up contributions over volume.

Retarded potentials.

extend this to time-dependent case.

guess form of solution:

For $\rho = \rho(\vec{r}, t)$, Φ must propagate out to observing

time taken $t_r = t - \frac{|\vec{r} - \vec{r}'|}{c}$ $\equiv$ "retarded time"

$\therefore$ claim that

$$\Phi(\vec{r}, t) = \int_V \frac{\rho(\vec{r}', t - \frac{|\vec{r} - \vec{r}'|}{c})}{|\vec{r} - \vec{r}'|} d^3r'$$

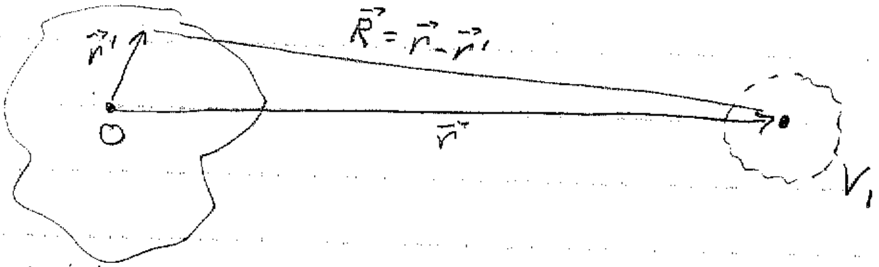
similar for $\vec{A}$

Sketch of proof:

separate volume of integration into two parts

$V = V_1 + V_2$

$\hookrightarrow$ nearby



w/in V_1, no retardation effects

$\Phi = \Phi_1 + \Phi_2$

$\Phi_1 = \int_{V_1} \rho(\vec{r}', t) / R \, d^3r'$ $\quad$ t not t_r

second part: evaluate Laplacian of Φ_2

$$\nabla^2 \Phi_2 = \int_{V_2} \nabla^2 \left(\frac{\rho(\vec{r}', t - R/c)}{R} \right) d^3r'$$

if we center our coord. system on obs. point, and hold $\vec{r}'$ fixed, then,

$$\frac{\rho(\vec{r}', t - R/c)}{R} \quad \text{is spherically symmetric}$$

$$\nabla^2 \left(\frac{\rho}{R}\right) \to \frac{1}{R} \frac{\partial^2 \rho}{\partial R^2}$$

since $f(t - R/c)$ satisfies $\frac{\partial^2 f}{\partial R^2} - \frac{1}{c^2} \frac{\partial^2 f}{\partial t^2} = 0$.

$$\nabla^2 (\rho/R) \to \frac{1}{R} \frac{1}{c^2} \partial_t^2 \rho$$

$$\nabla^2 \Phi_2 = \frac{1}{c^2} \frac{\partial}{\partial t^2} \underbrace{\int_{V_2} \frac{\rho(r', t - R/c)}{R} d^3 r'}_{\Phi_2}$$

$\therefore$ Φ_2 satisfies homogeneous wave eqn.

let $V_1 \to 0$ so that $\Phi_2 \approx \Phi$

$$\to \quad \nabla^2 (\Phi_1 + \Phi_2) = \frac{1}{c^2} \partial_t^2 \Phi - 4\pi \rho.$$

Φ_2 is the part that propagates to $\vec{r}$

Φ_1 is local contribution from ρ.

compact notation

$$\Phi(\vec{r}, t) = \int_V \frac{[\rho(\vec{r}')]}{R} d^3 r' \qquad R = |\vec{r} - \vec{r}'|$$

$[\;]$ = evaluate at $t_r = t - R/c$