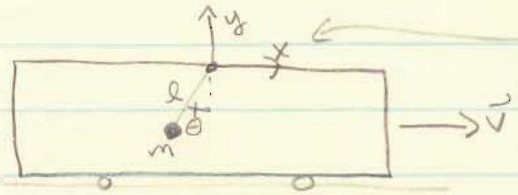
Test 2 solutions

axes only here at $t=\phi$.

1) a)



b) 3 rotational, 3 translational.

c) It's constrained to be in a plane (1 trans. constraint). $(z=\phi)$
It's stuck on the end of the end of the string (1 trans. constraint)
eqn: $x^2+y^2=l^2$ {at least at $t=0$}
it won't rotate at all except to maintain its relation to the string $\Rightarrow$ $\omega_z=\dot\theta$, $\omega_x=\omega_y=\phi$. If it's small, then the kin. energy from $\frac{1}{2}I_{cm}\omega_z^2$ is small. compared to $\frac{1}{2}mv^2$, so we neglect it {If you didn't get that that's ok}.

d) 1 left, $6-5=1$.

e) I choose θ, since I can easily represent T, U in that one var.

f) $T=\frac{1}{2}m(\dot x^2+\dot y^2)$; $x=vt-l\sin\theta$; $y=-l\cos\theta$

$\dot x = v - l\cos\theta\dot\theta$; $\dot y = l\sin\theta\dot\theta$

$$\rightarrow T=\frac{1}{2}m\left[(v-l\cos\theta\dot\theta)^2+(l\sin\theta\dot\theta)^2\right]$$

$$=\frac{1}{2}mv^2-mvl\cos\theta\dot\theta+\frac{1}{2}ml^2\dot\theta^2(\cos^2\theta+\sin^2\theta)$$

$$=\frac{1}{2}mv^2+\frac{1}{2}ml^2\dot\theta^2-mvl\cos\theta\dot\theta$$

$$U=+mgh=-mgl\cos\theta$$

$$\rightarrow \boxed{L=T-U=\frac{1}{2}mv^2+\frac{1}{2}ml^2\dot\theta^2-mvl\cos\theta\dot\theta+mgl\cos\theta}$$

(see next pg.)

Test 2 solns pg. 2

g) $H = \sum_j \frac{\partial L}{\partial \dot{q}_j}\dot{q}_j - L = \frac{\partial L}{\partial \dot{\theta}}\dot{\theta} - L$

$$H = (m\ell^2\dot{\theta} - \cancel{mv\ell\cos\theta})\dot{\theta} - \tfrac{1}{2}mv^2 - \tfrac{1}{2}m\ell^2\dot{\theta}^2 - \cancel{mv\ell\cos\theta\dot{\theta}} - mg\ell\cos\theta$$

$$\boxed{H = \tfrac{1}{2}m\ell^2\dot{\theta}^2 - \tfrac{1}{2}mv^2 - mg\ell\cos\theta}$$

h) Yes, because $\frac{dH}{dt} = \frac{\partial H}{\partial t} = \emptyset$

i) No, because $x(\theta, t)$ depends on t!

j) Transform to the reference frame of the train. In this ref. frame, $x = -\ell\sin\theta$ and $x(\theta, t)$ does not depend on t.
Note that the equations of motion for both situations must be the same.
You can't play this trick with an accelerating ref. frame because you have to start with a set of inertial cartesian coords.
Note that this shows that energy is not invariant to ref. frame transformations.

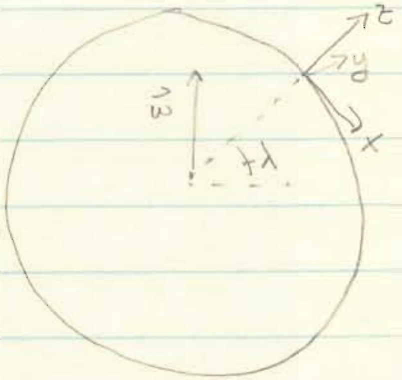
2) Easy way: $T = -\frac{1}{2}U$ in a circular orbit. $= -\frac{1}{2}\left(-\frac{GmM}{r}\right)$
By Virial Thm.

So if $M \to \frac{1}{2}M$, now $T_{after} = -U_{after}$ $\rightarrow$ $\boxed{E_{after} = \emptyset \text{ which is a parabolic orbit}}$
doesn't change; goes down by 1/2

Harder: You can show $T = -\frac{1}{2}U$ a bunch of diff. ways for a circular orbit. e.g. $F = \frac{GmM}{r^2} = ma_{cent} = \frac{mv^2}{r} \rightarrow mv^2 = \frac{GmM}{r} = -U = 2T$

a) It got rolled into $\vec{g}$.

b)



$$\vec{\omega} = \omega \sin\lambda \, \hat{e}_z - \omega\cos\lambda \, \hat{e}_x$$

$$a_z \sim -g \rightarrow v_z \sim -gt \quad \{v_{0z} = \emptyset\}$$

$$z \sim h - \tfrac{1}{2} g t^2$$

This amounts to neglecting the coriolis force. Now let's look, given that $\vec{v}_r = -gt\,\hat{e}_z$, what the Coriolis force is.

$$\vec{\omega}\times\vec{v}_r = (-\omega\cos\lambda)(-gt)\underbrace{(\hat{e}_x \times \hat{e}_z)}_{-\hat{e}_y} = -\omega g t \cos\lambda \, \hat{e}_y$$

$$\rightarrow \vec{a}_r \sim -g\hat{e}_z + 2\omega g t \cos\lambda \, \hat{e}_y .$$

$$\rightarrow \vec{v}_r \sim -gt\,\hat{e}_z + \omega g t^2 \cos\lambda \, \hat{e}_y$$

$\{$I used $\vec{v}_0 = \emptyset\}$

$$\rightarrow \vec{r} = \left(h - \tfrac{1}{2} g t^2\right)\hat{e}_z + \left(\tfrac{1}{3}\omega g t^3 \cos\lambda\right)\hat{e}_y$$

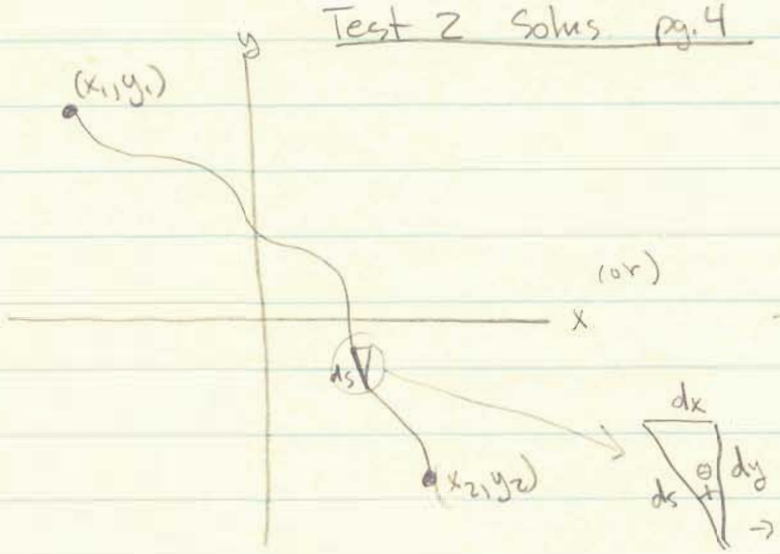
To get terms of higher order in ω, plug the new $\vec{v}_r$ into $\vec{\omega}\times\vec{v}_r$ and repeat. This would give a power series in ω. We'll just keep the first order term.

$$t_f = t(z=\emptyset) = \sqrt{\frac{2h}{g}}$$

$$\rightarrow y_f = \tfrac{1}{3}\omega g t_f^3 \cos\lambda = \frac{\omega g \cos\lambda}{3}\sqrt{\frac{8h^3}{g^3}} = \boxed{\frac{\omega\cos\lambda}{3}\sqrt{\frac{8h^3}{g}} = \text{deflection}}$$

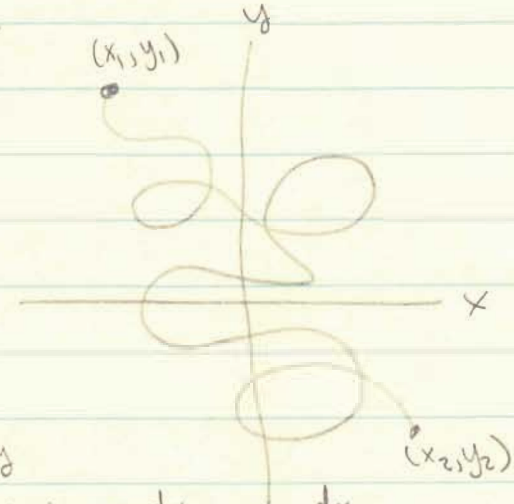
c) East $\{$my y direction$\}$. You could see this from cons. of ang. mom. as well.

4) a)

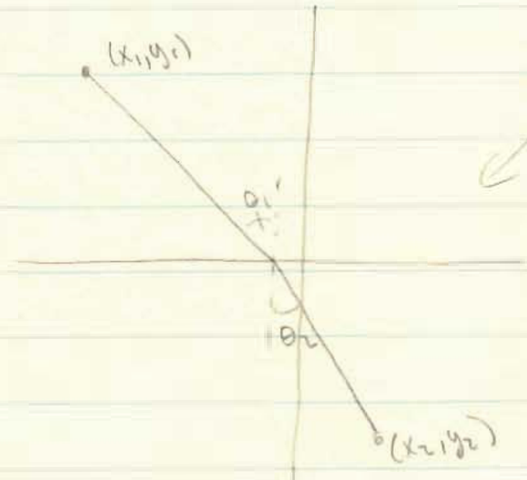


$$\rightarrow \sin\theta = \frac{dx}{ds} = \frac{dx}{\sqrt{dx^2+dy^2}}$$

(or)



(or)



I won't give full credit for this. It's assuming the solution.

b) $$t = \int_{t_1}^{t_2} dt = \int_{x_1}^{x_2} \frac{dt}{dx}\,dx = \int_{y_1}^{y_2} \frac{dt}{dy}\,dy = \int_0^{s_f} \frac{dt}{ds}\,ds = \int_0^{s_f} \frac{1}{v}\,ds = \int_0^{s_f} \frac{n}{c}\,ds$$

$$ds = \sqrt{dx^2+dy^2} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

$$\rightarrow \boxed{t = \int_{x_1}^{x_2} \frac{n\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{c}\,dx = \int_{y_1}^{y_2} \frac{n\sqrt{1+\left(\frac{dx}{dy}\right)^2}}{c}\,dy}$$

c) Now n is an explicit function of y → use dy equation because $\frac{\partial n}{\partial x} = \emptyset$.

$$f = \frac{n}{c}\sqrt{1+x'^2} \rightarrow \underbrace{\frac{\partial f}{\partial x}}_{\emptyset} - \frac{d}{dy}\left(\frac{\partial f}{\partial x'}\right) = \emptyset \rightarrow \frac{\partial f}{\partial x'} = \text{const}$$

$$\rightarrow \frac{n}{c}\frac{x'}{\sqrt{1+x'^2}} = \text{const} = \frac{n}{c}\frac{dx/dy}{\sqrt{1+\left(\frac{dx}{dy}\right)^2}} = \frac{n}{c}\frac{dx}{\sqrt{dx^2+dy^2}} = \frac{n}{c}\sin\theta = \text{const}$$

or $\boxed{n\sin\theta = \text{const}}$