Lecture 15

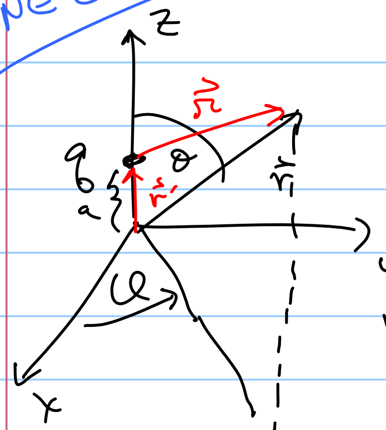
Note Title 2/17/2006

Two types of electrostatics problems

- Bndry value problem (sep. variables)
- Summation problem

$$V = \frac{1}{4\pi\epsilon_0}\int \frac{\rho\, d\tau}{\imath} \quad \text{or} \quad \vec{E} = \frac{1}{4\pi\epsilon_0}\int \frac{\rho\,\hat{\imath}\, d\tau}{\imath^2}$$

ONE CHARGE



$$V(r,\theta) = \frac{1}{4\pi\epsilon_0}\frac{q}{\imath} = \frac{q}{4\pi\epsilon_0}\frac{1}{(r^2+a^2-2ar\cos\theta)^{1/2}}$$

Assume $r > a$ and expand in small parameter $\frac{a^2}{r^2} - \frac{2a}{r}\cos\theta = \epsilon$

↖Taylor or binomial expansion

$$V(r,\theta) \propto \frac{1}{\imath} = \frac{1}{r(1+\epsilon)^{1/2}} = \frac{1}{r}\left[1 - \frac{1}{2}\underbrace{\left(\frac{a^2}{r^2} - \frac{2a}{r}\cos\theta\right)}_{\epsilon} + \frac{3}{8}\left(\frac{a^2}{r^2} - \frac{2a}{r}\cos\theta\right)^2 + \dots\right]$$

Now arrange in powers of $\frac{a}{r} \ll 1$; $\quad \frac{1}{\imath} = \frac{1}{r}\left[1 + \left(\frac{a}{r}\right)\cos\theta + \left(\frac{a}{r}\right)^2\left[\frac{3\cos^2\theta - 1}{2}\right] + \dots\right]$

$$\boxed{\frac{1}{\imath} = \frac{1}{r}\sum_{\ell=0}^{\infty}\left(\frac{a}{r}\right)^{\ell} P_\ell(\cos\theta) \qquad r > a}$$

Note that for $a << r$ V is due to a point charge even though q is not at $\vec{r}' = 0$. Note that for $a < r$ V is a complicated function & the series converges slowly.

So why would we want to do this complicated expansion? It turns out we will use only the 2^{nd} term in the expansion, since the other higher order terms will be neglible in the following chapters.

Ex: $\boxed{\text{Axial multipole expansion}}$

assume $\rho(\vec{r}') = \delta(x')\,\delta(y')\,\lambda(z')$ ← either continuous or delta functions

$$V = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}')\,d^3r'}{r} = \frac{1}{4\pi\epsilon_0}\iiint \delta(x')\,dx'\,\delta(y')\,dy'\,\frac{\lambda(z')\,dz'}{(r^2+z'^2-2z'r\cos\theta)^{1/2}}$$

$$= \frac{1}{4\pi\epsilon_0}\int \frac{\lambda(z')}{r}\sum_{l=0}^{\infty}\left(\frac{z'}{r}\right)^l P_l(\cos\theta)\,dz' = \frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{M_l\,P_l(\cos\theta)}{r^{l+1}}$$

where $M_l = \int \lambda(z')(z')^l\,dz'$ are the axial multipole moments!

$M_0 = \int \lambda(z')\,dz'$ is the monopole moment $= Q_{total}$

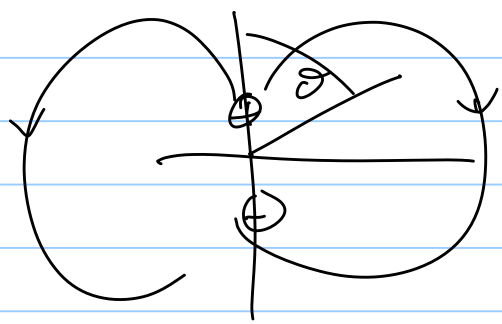
M_1 is the dipole moment. Note $V \propto \frac{1}{r^2}$ since the dipole contains a + & − charge.

example: $\lambda(z') = q_0\,\delta(z' - \frac{a}{2}) - q_0\,\delta(z' + \frac{a}{2})$

$$M_\ell = \int q_0\,\delta(z' - \tfrac{a}{2})(z')^\ell dz' - \int q_0\,\delta(z' + \tfrac{a}{2})(z')^\ell dz' = \begin{cases} 0 & \ell = 0 \\ \left(\frac{a}{2}\right)^\ell 2q_0 & \ell = 1, 3, 5, \ldots \end{cases}$$

$$V_{dipole} = \frac{1}{4\pi\epsilon_0}\frac{aq_0\cos\vartheta}{r^2} \qquad \vec{E} = -\vec{\nabla} V_{dipole}$$

spherical $\hat{r}, \hat{\vartheta}, \hat{\varphi}$

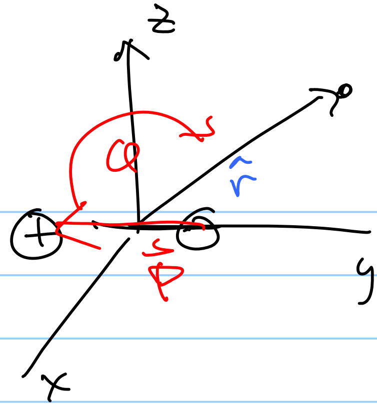


field lines

$\oplus \leftarrow \ominus$ $= \vec{p}$

$\vec{p}$ dipole moment $\rightarrow$ $\boxed{V = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2}}$

$|\vec{p}| = qa$

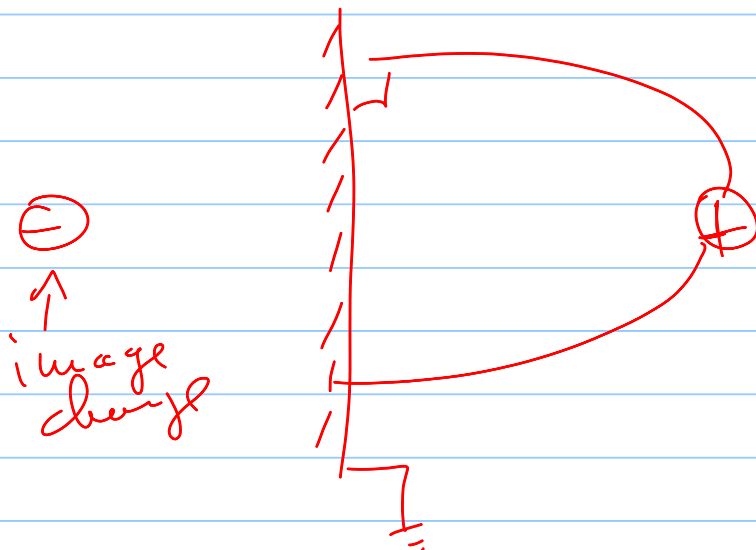


$$\vec{p}\cdot\hat{r} = |\vec{p}||\hat{r}|\cos\varphi$$

$|\hat{r}| = 1$

$\varphi = 90+\theta$

$$V = \frac{1}{4\pi\epsilon_0}|\vec{p}|\frac{\cos(90+\theta)}{r^2}$$



$E_{\parallel} = 0$

conductor

no gravity