

Guided-waves

consider a wave propagating in the z direction
confined in x, y

linear eqn:

$$\nabla^2 E + \frac{n^2(\vec{p})\omega^2}{c^2} E = 0$$

$$\vec{p} = x\hat{x} + y\hat{y}$$

$n(\vec{p})$ is transverse index

Assume we can write

$$E(\vec{p}, z) = A(\vec{p}) e^{ik_z z}$$

profile of waveguide.

k_z = propagation constant
(sometimes written β)

$$\rightarrow \nabla_{\perp}^2 A + \left(\frac{\omega^2 n^2(\vec{p})}{c^2} - k_z^2 \right) A = 0$$

This is an eigenvalue equation:

$$\nabla_{\perp}^2 U_m + \frac{\omega^2 n^2(\vec{p})}{c^2} U_m = k_{zm}^2 U_m$$

compare to QM:

$$-\frac{\hbar^2}{2m} \nabla^2 \psi + V(r) \psi = E \psi$$

$$\text{so } -\frac{\omega^2 n^2(\vec{p})}{c^2} \sim V(r) \quad \text{effective pot'}$$

$$-k_z^2 \sim E \quad \text{eigenvalue.}$$

step index

QM finite square well



range of bound k_z :

$$\frac{\omega n_1}{c} < k_{zm} < \frac{\omega n_2}{c}$$

unbound $k_{zm} > \frac{\omega n_2}{c}$ = radiation modes



range of bound states:

$$0 < E_n < V_0$$

$\frac{k_{zm}}{(\omega/c)}$ = effective index of mode.

Simple example: ideal metallic waveguide
if $E \rightarrow 0$ at walls, effective $V(x)$ is like ∞ sq. well

for 1D slab waveguide, walls at $x=0, L$ vacuum

$$U_m(x) = C \sin\left(\frac{m\pi}{L}x\right) \quad \text{inside}$$

$$\text{normalize s.t. } \int |U_m(x)|^2 dx = 1 \quad C = \sqrt{\frac{C}{L}}$$

$$\frac{d^2 U_m}{dx^2} = -\left(\frac{m\pi}{L}\right)^2 U_m = -\frac{\omega^2}{c^2} U_m + k_{zm}^2 U_m$$

$$\rightarrow \frac{\omega^2}{c^2} = \left(\frac{m\pi}{L}\right)^2 + k_{zm}^2 \quad \text{modal dispersion relation.}$$

LHS = k_z for plane wave in central region = k_0

$$\frac{m\pi}{L} = k_x$$

so dispersion relation reads $k^2 = k_x^2 + k_z^2$

this is true in general.

$$\text{since } k_{zm} = \sqrt{k^2 - \left(\frac{m\pi}{L}\right)^2}$$

$$k_{zm} < k_0 \text{ and } (V_{ph})_m = \frac{\omega}{k_{zm}} > \frac{\omega}{k_0}$$

- modal phase velocity is faster than it would be in that medium if unguided

- wavefronts of individ. modes are flat.

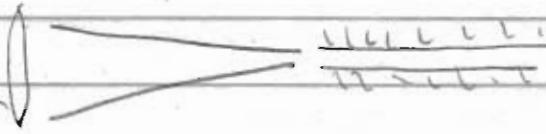
$$2D \text{ eq. } U_{mn}(x, y) = \frac{1}{L_y} \sin\left(\frac{m\pi}{L_x} x\right) \sin\left(\frac{n\pi}{L_y} y\right)$$

$$(k_{zm})_{mn} = \sqrt{k_0^2 - k_x^2 - k_y^2}$$

actual modes must satisfy EM boundary conditions.

for large m, n $k_z \rightarrow \text{imag}$. These modes are cutoff, expn. damped

Coupling of arbitrary input to waveguide.



input field $E_{in}(x, y, z=0)$
at $z=0$

- if only bound modes can propagate for $z > 0$
then $U_m(\vec{p})$ form a complete set, orthogonal

$$E_{in}(x, y) = \sum_m a_m U_m(\vec{p}) \quad \left\{ U_n^* U_m d\vec{p} = 0 \text{ for } n \neq m \right.$$

$$a_m = \int U_m^* E_{in} dx dy d\vec{p}$$

- projection onto cutoff modes \rightarrow reflected power.

$$\alpha_m = ik_{zm} = \sqrt{k_x^2 - k_z^2}$$

inside w.g. mode is exponentially damped.

e.g.

$$\sin\left(\frac{m\pi}{L}x\right) e^{-\alpha_m x}$$

power flow is reflected back,

- note that spatial phase of input affects coupling,

e.g. $E_{in} = U_m(\vec{p}) e^{ik_x x}$ (angled but right shape)

$$a_m = \int_0^L |U_m(x)|^2 e^{ik_x x} dx \neq 1$$

- for dielectric w.g. unguided modes are radiated away.

Evolution of field and intensity with z .

$$E(x, y, z) = E_0 \sum_{mn} a_{mn} U_{mn}(x, y) e^{i\beta_{mn} z}$$

e.g. 2 modes

$$E = E_0 \sqrt{c} (a_1 \sin(\frac{\pi}{L} x) e^{i\beta_1 z} + a_2 \sin(\frac{2\pi}{L} x) e^{i\beta_2 z})$$

note that relative phase evolves.

intensity

$$I \propto |E|^2 = E_0^2 (|a_1|^2 |U_1(x)|^2 + |a_2|^2 |U_2(x)|^2 + a_1 a_2^* U_1 U_2^* e^{i(\beta_1 - \beta_2)z} + c.c.)$$

cross term leads to mode beating.

Conceptually, method is like Fresnel propagation:

decompose initial field into modal spectrum a_m

evolve phases of modes $a_m e^{i\beta_m z}$

spatial field = weighted sum of modes

Nonlinear guided wave propagation

- parametric mixing: $\chi^{(2)}$ or $\chi^{(3)}$ freq. conversion
- NL propagation accounting for n_2

Parametric mixing / harmonic generation

start with general coupled equations: (see 2.1, 2.3)

$$\nabla^2 \vec{E}_n(\vec{r}) + \frac{\omega_n^2 \epsilon^{(1)}(\omega_n)}{c^2} \cdot \vec{E}_n(\vec{r}) = - \frac{\omega_n^2}{c^2} \vec{P}_n^{NL}(\vec{r})$$

assume no birefringence, $\vec{\epsilon}^{(1)}(\omega_n) \rightarrow \eta^2(\vec{r}, \omega_n)$

assume all waves share same linear pol. state.

LHS is linear wave eqn

- mode solutions: $U_m(r, \omega)$ in general ω -dependent
 $B_m(\omega)$

Specific example: $\omega_3 = 3\omega_1$, 3rd harmonic

input ω_1 in mode m

for non-depleted pump appx. write eqn for ω_3 :

$$\nabla^2 E_3 + \frac{\omega_3^2 \eta^2(r, \omega)}{c^2} E_3 = - \frac{\omega_3^2}{c^2} \chi^{(3)} F_{1, \text{in}} e^{i 3 \beta_1 z}$$

For simplicity assume $U_m(r)$ is indep of ω

(this is oppx the case for hollow waveguides)

write E_3 as sum over modes

$$E_3 = \sum_m a_m(z) U_m(r) e^{i \beta_m z}$$

$a_m(z)$ varies slowly

LHS:

$$\begin{aligned} &= \sum_{m'} \left(a_{ml}(z) \left(\nabla_r^2 u_{m'} + \frac{w^2 n^2(r)}{c^2} u_{m'} - \beta_{m'}^2 u_{m'} \right) e^{i\beta_{m'} z} \right. \\ &\quad \left. + (i\beta_{m'} \partial_z a_{ml} + \partial_z^2 a_{ml}) u_{m'} \right) e^{i\beta_{m'} z} \\ &= \sum_{m'} i u_{m'} \beta_{m'} \partial_z a_{ml} e^{i\beta_{m'} z} = - \frac{w^2}{c^2} \chi^{(3)} E_1^{(3)}(r) e^{i(3\beta_l - \beta_m)z} \end{aligned}$$

spatially RHS is not just one mode

get eqn. for coeff of one mode, $a_{ll}(z)$
mult thru by $u_l^*(r)$, integrate.

$$i \beta_l \partial_z a_{ll} e^{i\beta_l z} = "$$

$$\rightarrow \partial_z a_{ll} = -i \frac{w^2}{\beta_l c^2} \chi^{(3)} \left\{ u_l^*(r) E_1^{(3)}(r) d^3 r e^{i(3\beta_l - \beta_{3l})z} \right\}$$

notice: $e^{i(3\beta_l - \beta_{3l})z} \rightarrow$ phase mismatch $\Delta\beta$
this is affected by the mode.

can use for phase matching.

- typically higher $w \rightarrow$ higher $n(w)$
- go to high mode l to compensate
match effective index

integral \rightarrow mode overlap

how well does induced P^{NC} overlap with mode l

Efficiency: $\Delta\beta = 0$ and good $\left\{ u_l^*(r) u_l^{(3)}(r) d^3 r \right\}$

Gridded waves + NL index

intensity changes refractive index profile

$$n(r) = n_0(r) + n_2 I(r)$$

$$\nabla_T^2 U_m + \frac{\omega^2}{c^2} (n_0(r) + n_2 I(r))^2 U_m = \beta_m^2 U_m$$

$$\text{if } n_2 I \ll n_0, \quad n^2(r) \approx n_0^2(r) + 2n_0(r)n_2 I(r)$$

In general, eigenmodes will change.

For small $n_2 I$, use perturbation theory

$$\text{in QM: 1st order energy shift } \Delta E_n = \int \psi_n^* V'(r) \psi_n d^3 r$$

$$\text{here, our perturbation is } \frac{\omega^2 \cdot 2n_0(r)n_2 I(r)}{c^2}$$

$$\text{and the eigenvalue shift is } \Delta \beta_m = (\beta_m + \Delta \beta_m)^2 - \beta_m^2 \\ \approx 2\beta_m \Delta \beta_m$$

$$2\beta_m \Delta \beta_m = 2 \frac{\omega^2 n_2}{c^2} \int |U_m(r)|^2 n_0(r) I(r) d^3 r$$

$$\Delta \beta_m = \frac{\omega n_2 I_0}{c (n_{eff})_m} \int |U_m(r)|^4 n_0(r) d^3 r$$

$$\text{where } (n_{eff})_m = \beta_m / (\omega/c) = \text{effective index of mode } m$$

this assumes the mode $U_m(r)$ function is normalized

if not,

$$\Delta \beta_m = \frac{\omega n_2}{c (n_{eff})_m} \int |U_m(r)|^2 n_0(r) P_0 |U_m(r)|^2 d^3 r$$

$$= \text{mode, index weighted avg I} \left[\int |U_m(r)|^2 d^3 r \right]^2$$

We can define the effective mode area

$$A_{\text{eff}} = (n_{\text{eff}}) \frac{\int [U_m(r)]^2 d^2 r}{\int [U_m(r)]^4 n_o(r) d^2 r}$$

$$\text{then } \Delta \beta_m = \frac{w_0 n_2}{c A_{\text{eff}}} P$$

$\underbrace{}$
 γ

↳ a measure of the nonlinearity strength.

To 1st order the NL phase shift is applied equally across the mode

There is some power coupled to the other modes

start in mode m , couple to mode l

the new mode has contributions from other modes

$$Q_m: \Psi^{(1)} = \Psi^{(0)} + \sum_{m' \neq m} \frac{|\Delta E_{ml}|^2}{E_q - E_m} \Psi^{(0)}$$

$$U_m^{(1)}(r) = U_m^{(0)}(r) + \sum_{\substack{n'm' \\ \neq nm}} \frac{(\Delta \beta_{mn})^2}{\beta_m^2 - \beta_{m'n'}^2} U_{m'n'}^{(0)}(r)$$

the NL index \rightarrow transfer of power to other modes

can set up coupled diff. eqns for each mode with NL overlap as the source term.

Note that the perturbation may have some symmetry that prevents coupling to certain modes.