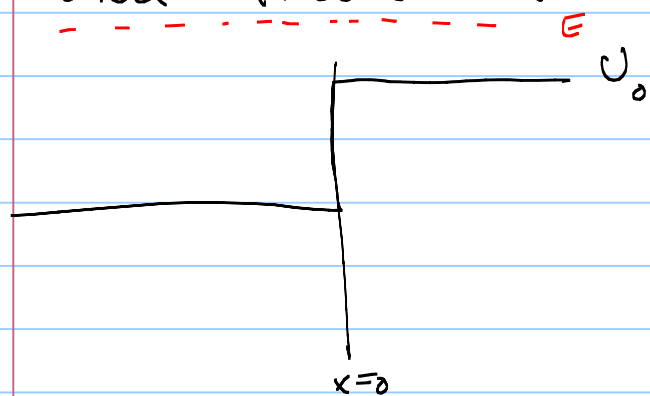
exam 2 sample questions : solutions

Note Title 2/27/2008

1) A particle is incident from left with energy $E > U_0$. Compute the reflection and transmission coeff.



$x < 0$ $\qquad \psi'' = -\frac{2mE}{\hbar^2}\psi \Rightarrow \psi'' + k^2\psi = 0$

$$\sqrt{2mE}/\hbar = k$$

$$\psi(x) = Ae^{ikx} + Be^{-ikx}$$

$x > 0.$ $\qquad \psi'' = -\frac{2m}{\hbar^2}(E - U_0)\psi \Rightarrow \psi'' + \ell^2\psi = 0$

$$\ell = \frac{\sqrt{2m(E-U_0)}}{\hbar}$$

$$\psi = Fe^{i\ell x}$$

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cont. of ψ : $\quad A + B = F$

ψ' : $\quad ik(A - B) = i\ell F$

$$\Rightarrow \quad A + B = \frac{k}{\ell}(A - B)$$

$$B(1 + k/\ell) = \left(\frac{k}{\ell} - 1\right)A$$

$$B/A = \frac{k}{e} - 1 \Big/ 1 + k/e \Rightarrow \boxed{\left(\frac{k-e}{e+k}\right)^2 = R}$$

$$B = F - A \quad , \quad \frac{-\cancel{i}eF + ikA}{\cancel{i}k} = B$$

$$F - A = A - \frac{e}{k}F \Rightarrow F\left(1 + \frac{e}{k}\right) = 2A$$

$$\Rightarrow F(k+e) = 2Ak$$

$$\Rightarrow \frac{F}{A} = \frac{2k}{k+e}$$

$T + R = 1$

$$\Rightarrow \boxed{T = 4k^2/(k+e)^2}$$

$$\frac{(k-e)^2}{(e+k)^2} + \frac{4k^2}{(e+k)^2} = \frac{k^2+2ke+e^2}{(e+k)^2} = \frac{(k+e)^2}{(k+e)^2} = 1$$

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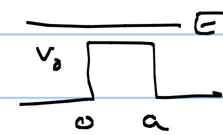
2) See the Scan on the wiki within the questions

3) See 3/3/08 lecture notes for setup.



I $\quad A e^{ik_1 x} + B e^{-ik_1 x}$

II $\quad C e^{ik_2 x} + D e^{-ik_2 x}$

III $\quad F e^{ik_1 x}$

cont. of ψ: a) $A + B = C + D$

b) $C e^{ik_2 a} + D e^{-ik_2 a} = F e^{ik_1 a}$

ψ' c) $ik_1 (A - B) = ik_2 (C - D)$

d) $ik_2 (C e^{ik_2 a} - D e^{-ik_2 a}) = ik_1 F e^{ik_1 a}$

a) $\Rightarrow$ $A + B - C = D$

c) $\Rightarrow$ $-\frac{k_1}{k_2}(A - B) + C = D$

$\Rightarrow$ $D = A + B - \frac{k_1}{k_2}(A - B)$

$C = \frac{1}{2}\left[A + B + \frac{k_1}{k_2}(A - B)\right]$

This all you need. 4 equations in 5 unkn.

Now use these to replace C + D in b) + d)

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$$\frac{k_2}{k_1}\left[C e^{ik_2 a} - D e^{-ik_2 a}\right] = C e^{ik_2 a} - D e^{-ik_2 a}$$

$$D = A + B - \frac{k_1}{k_2}(A - B) = A\left[1 - k_1/k_2\right] + B\left[1 + k_1/k_2\right]$$

$$C = \frac{1}{2}\left[A + B + \frac{k_1}{k_2}(A - B)\right] = \frac{1}{2}A\left[1 + \frac{k_1}{k_2}\right] + \frac{1}{2}B\left[1 - k_1/k_2\right]$$

$$\Rightarrow \left(\frac{k_2}{k_1} - 1\right) C e^{ik_2 a} = \left(\frac{k_2}{k_1} + 1\right) D e^{-ik_2 a}$$

$$\left(\frac{k_2}{k_1} - 1\right) e^{ik_2 a}\left[\frac{1}{2}A\left[1 + \frac{k_1}{k_2}\right] + \frac{1}{2}B\left[1 - k_1/k_2\right]\right]$$

$$= \left(\frac{k_2}{k_1} + 1\right) e^{-ik_2 a}\left[A\left[1 - k_1/k_2\right] + B\left[1 + k_1/k_2\right]\right]$$

$$(k_2 - k_1) e^{2ik_2 a}\left[\frac{1}{2}A\left[1 + k_1/k_2\right] + \frac{1}{2}B\left[1 - k_1/k_2\right]\right]$$

$$= (k_2 + k_1)\left[A\left[1 - k_1/k_2\right] + B\left[1 + k_1/k_2\right]\right]$$

$$A\left\{(k_2 - k_1) e^{2ik_2 a}\,\frac{1}{2}(1 + k_1/k_2) - (k_2 + k_1)(1 - k_1/k_2)\right\}$$

$$= B\left\{-(k_2 - k_1) e^{2ik_2 a}\,\frac{1}{2}(1 - k_1/k_2) + (k_1 + k_2)(1 + k_1/k_2)\right\}$$

$$\Rightarrow A\left\{\frac{e^{2ik_2 a}}{2}(k_2 - k_1)(k_2 + k_1) - (k_2 + k_1)(k_2 - k_1)\right\}$$

$$= B\left\{-\frac{e^{2ik_2 a}}{2}(k_2 - k_1)(k_2 - k_1) + (k_1 + k_2)(k_2 + k_1)\right\}$$

This allows you to write B/A.
Similar argument for P/A.

4) $E_N = \alpha N^2$

$E_{N+1} = \alpha (N+1)^2 \Rightarrow E_{N+1} - E_N = \alpha (N^2 + 2N + 1) - \alpha N^2$

$$\Rightarrow \frac{E_{N+1} - E_N}{E_N} = \frac{2N+1}{N^2}$$

5) $a^+ = \sqrt{\frac{m\omega}{2\hbar}} \left(x - \frac{i}{m\omega} p \right)$
$= \sqrt{\frac{\alpha}{2}} \left(x - \frac{i}{m\omega} \left(-i\hbar \frac{d}{dx} \right) \right)$

$\psi_0(x) = \left(\frac{\alpha}{\pi} \right)^{1/4} e^{-y^2/2} \qquad y = \sqrt{\alpha}\, x$

apply a^+ to ψ_0 and normalize to get

$$a^+ \psi_0 = \left(\frac{\alpha}{\pi} \right)^{1/4} \sqrt{2}\, y\, e^{-y^2/2}$$

plug this into Schrodinger with the potential $V(x) = \frac{1}{2} m \omega^2 x^2$ and you will get $E = \frac{3}{2} \hbar \omega$.

6) $S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}$

$$S^{+}S = \begin{pmatrix} S_{11}^{*} & S_{21}^{*} \\ S_{12}^{*} & S_{22}^{*} \end{pmatrix}\begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} = \begin{pmatrix} |S_{11}|^2 + |S_{21}|^2 & S_{11}^{*}S_{12} + S_{21}^{*}S_{22} \\ S_{12}^{*}S_{11} + S_{22}^{*}S_{21} & |S_{12}|^2 + |S_{22}|^2 \end{pmatrix}$$

$S^{+}S = I \Rightarrow$ 1) $|S_{11}|^2 + |S_{21}|^2 = 1 = T_e + R_e$

2) $|S_{12}|^2 + |S_{22}|^2 = 1 = T_R + R_r$

3) $S_{12}^{*} S_{11} + S_{22}^{*} S_{21} = 0$

4) $S_{11}^{*} S_{12} + S_{21}^{*} S_{22} = 0$

1) + 2) are energy conservation

7) $E = \hbar \omega = hf = 6.6 \times 10^{-34}\ JS \cdot 10^{14}\ \frac{1}{S}$

$= 6.6 \times 10^{-20}\ J$

$6.6 \times 10^{-20}\ J \cdot 6 \times 10^{18}\ eV/J \approx .4eV$

8) $a^{+}a = \frac{1}{\hbar\omega} H - \frac{1}{2}$ (2.54 in Book)

$$\underbrace{a^{+}a\,\psi_N} = \frac{1}{\hbar\omega}\overbrace{H\psi_N}^{(N+\frac{1}{2})\hbar\omega\,\psi_N} - \frac{1}{2}\psi_N$$

$$= \left(N + \tfrac{1}{2}\right)\psi_N - \tfrac{1}{2}\psi_N$$

$$= \underbrace{N\psi_N}$$

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9) $\psi(x) = \alpha x e^{-\beta x} e^{i\gamma t/\hbar}$

$$\psi'(x) = \alpha e^{-\beta x} - \alpha\beta x e^{-\beta x}$$

$$\psi''(x) = \underbrace{-\alpha\beta e^{-\beta x} - \alpha\beta e^{-\beta x}} + \alpha\beta^2 x e^{-\beta x}$$

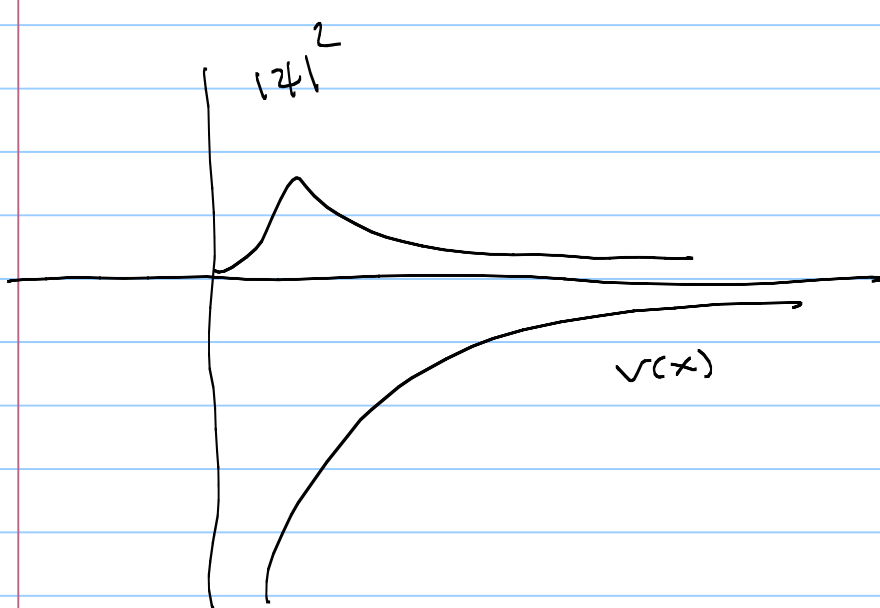
$$= -2\alpha\beta e^{-\beta x} + \alpha\beta^2 x e^{-\beta x}$$

$$\Rightarrow -\psi''(x) = \left(\frac{2\beta}{x} - \beta^2\right)\alpha x e^{-\beta x}$$

Schröedinger ($\hbar^2/2m = 1$)

$$-\psi'' = (E - V_0)\psi$$

So if we put $V(x) = -\dfrac{2\beta}{x}$, $E = -\beta^2$



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