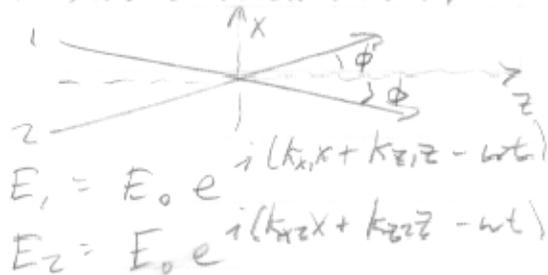


Scalar approach to guided waves

- assume simple boundary conditions by ignoring vector character of waves
- can still get basic forms of waves, v_{ph} , v_{gr} , cutoff

1) Start: two crossed beams



$$E_1 = E_0 e^{i(k_{x1}x + k_{z1}z - \omega t)}$$

$$E_2 = E_0 e^{i(k_{x2}x + k_{z2}z - \omega t)}$$

$$k_{x1} = -k_0 n \sin \phi \quad k_{z1} = k_0 n \cos \phi$$

$$k_{x2} = +k_0 n \sin \phi \quad k_{z2} = k_{z1}$$

$$E_{tot} = E_1 + E_2 = E_0 e^{i(k_{z2}z - \omega t)} \left(e^{i k_0 n \sin \phi x} + e^{-i k_0 n \sin \phi x} \right)$$

$$= 2 E_0 e^{i(k_{z2}z - \omega t)} \underbrace{\cos(k_0 n \sin \phi x)}_{\substack{\text{traveling} \\ \text{wave}}} \rightarrow \text{standing wave} \rightarrow \text{transverse envelope on amplitude.}$$

if we place a screen \parallel to x-y plane, intensity will show fringes

$$I \propto |E|^2 = 4 E_0^2 \cos^2(k_x x)$$

with $k_x = k_0 n \sin \phi$

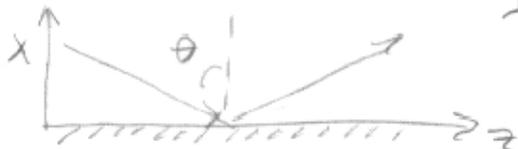
these are interference fringes

- node factor of 4 $I_{peak} = 4 I_0$

- fringe spacing depends on crossing angle. (and n, λ)



2) Reflected wave: same thing



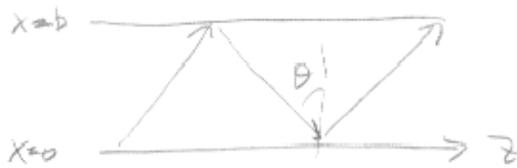
$$E_{tot} = E_0 e^{i(k_{z2}z - \omega t)} \left(e^{i k_x x} - e^{-i k_x x} \right)$$

$$= 2i E_0 e^{i(k_{z2}z - \omega t)} \sin k_x x \quad \text{to ensure } E(x=0) = 0$$

$$k_x = k_0 n \cos \theta$$

Same standing wave structure, just placed metal surface at node.

3) waveguide



same solution for $0 \leq x \leq b$
 but now choose b (or x, n, θ)
 so that $\sin(k_x b) = 0$
 $\rightarrow k_x = m\pi/b$
 $m = \text{integer} \geq 1$

this is a quantization condition on k_x
 very similar to QM \rightarrow square well
 \rightarrow discrete modes.



Waveguide has spatial modes $m = \text{mode index}$.

Traveling wave component: $e^{i(k_z z - \omega t)}$

through wave eqn,

$$k_x^2 + k_z^2 = k_0^2 n^2$$

$$\rightarrow k_z = \sqrt{k_0^2 n^2 - k_x^2} = \sqrt{\frac{\omega^2 n^2}{c^2} - \frac{m^2 \pi^2}{b^2}}$$

given $\omega, n, b \rightarrow$ discrete values for k_z

$k_z \equiv$ propagation constant.

\rightarrow each mode has a planar wavefront

\rightarrow propagating at $v_{ph} = \omega / k_z$
 $k_z \neq k_x$

since k_z depends on m ,
 \rightarrow "modal dispersion"

look at case where $n=1$

free wave $v_{ph} = \frac{\omega}{k_0} = c$

guided wave $v_{ph} = \frac{\omega}{\sqrt{\frac{\omega^2}{c^2} - \frac{m^2 \pi^2}{b^2}}} = \frac{c}{\sqrt{1 - \frac{m^2 \pi^2 c^2}{\omega^2 b^2}}} > c!$

$V_{ph} > c$ because of angle of waves to z -axis

group velocity: $V_{gr} = \frac{d\omega}{dk_z}$

either solve for $\omega(k_z)$ or calculate

$$V_{gr} = \left(\frac{dk_z}{d\omega} \right)^{-1}$$

this gives $V_{gr} < c$

each mode propagates at diff't $V_{ph} \rightarrow$ "mode beating" = interference btw. diff't propagating modes

each mode prop. at diff't $V_{gr} \rightarrow$ walkoff of pulses

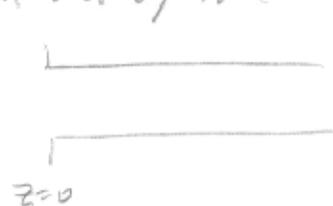
- initial pulse coupled to two modes

- at end of waveguide pulses emerge at diff't times

$$z_{m=1} = L / V_{gr}(m=1)$$

$$z_{m=2} = L / V_{gr}(m=2)$$

Coupling to a waveguide:

 if $E_{in}(x, z=0)$ input wave matches $E(x)$ for a given m , wave will be launched.

(actual calculation: if $U_m(x) = A_m \sin\left(\frac{m\pi}{b}x\right)$ = transv. mode expand input in series

$$E_{inside} = \sum \alpha_m U_m(x)$$

$$\text{w/ } \alpha_m = \int_0^b U_m^*(x) E_{in}(x) dx = \text{overlap integral}$$

Cutoff condition:

As we decrease ω , reach a point where $k_z = 0$

$$k_z = \sqrt{\left(\frac{\omega c n}{b}\right)^2 - \left(\frac{m\pi}{b}\right)^2} = 0 \quad \text{defines cutoff frequency.}$$

Suppose $m=1$ and $\omega < \omega_{c1} = \frac{\pi c}{b}$

then $k_z = i|k_z|$ pure imag.

→ only evanescent wave is allowed in w.g.
input wave is reflected.



Each mode has its own cutoff

if $\omega_{c1} < \omega < \omega_{c2}$ only $m=1$ propagates.

→ "single-mode waveguide"

Extension to 2-D walls



$$k_x = \frac{m_x \pi}{b_x} \quad k_y = \frac{m_y \pi}{b_y} \quad k_z = \sqrt{k_0^2 n^2 - \pi^2 \left(\frac{m_x^2}{b_x^2} + \frac{m_y^2}{b_y^2} \right)}$$

standing wave in x, y $\sin(k_x x + k_y y)$.

still propagates in z , range of allowed ω 's

Extend to 3-D box: resonance

$$\text{now } k_z = m_z \pi / b_z$$

only discrete ω 's allowed!