either by considering $\lim_{\Delta t \to 0} [N(t + \Delta t) - N(t)]/\Delta t$ or by using the chain rule for derivatives.

(b) Using $\partial \rho/\partial t = -\partial/\partial x\,(\rho u)$ show that

$$\frac{dN}{dt} = -\rho(b, t)\left[u(b, t) - \frac{db}{dt}\right] + \rho(a, t)u(a, t).$$

(c) Interpret the result of part (b) if the moving observer is in a car moving with the traffic.

60.3. (a) Without using any mathematics, explain why $\int_{a(t)}^{b(t)} \rho(x, t)\, dx$ is constant if a and b (not equal to each other) both move with the traffic.

(b) Using part (a), rederive equation 60.2.

(c) Assuming $\partial \rho/\partial t = -(\partial/\partial x)(\rho u)$, verify mathematically that part (a) is valid (exercise 60.2 may be helpful).

60.4. If the traffic flow is increasing as x increases ($\partial q/\partial x > 0$), explain physically why the density must be decreasing in time ($\partial \rho/\partial t < 0$).

61. A Velocity-Density Relationship

The two variables, traffic density and car velocity, are related by only one equation, conservation of cars,

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0.} \qquad (61.1)$$

If the velocity field is known, equation 61.1 reduces to a partial differential equation for the unknown traffic density. In this case, equation 61.1 can be used to predict the future traffic density if the initial traffic density is known. As an initial value problem, equation 61.1 is analogous to the ordinary differential equation for the position of a mass or the ordinary differential equations developed in population dynamics. We could pursue the solving of this partial differential equation. However, this appears senseless since *we do not know the velocity field.*

The unknown velocity field must be investigated. Considering the cars as particles, we need to know the particles' velocities. If this was a mechanical system, then we would investigate the forces in the system and use Newton's law to study the motion of the particles. However, there is no equivalent to Newton's law prescribing the manner in which cars must move. It is not forces which cause cars to move; it is the decisions of individual drivers. What factors influence individual car velocities?

A way to investigate the velocity field is to conduct experiments to understand an individual's response to traffic stimuli. Why does a particular driver drive the way he or she does? We will postpone a discussion of such experiments and the resulting mathematical models (see Sec. 64 on car-following models). Instead, let us describe some observed traffic phenomena with which you are no doubt familar. If traffic is sufficiently light, then the driver of each car has the freedom to do as he or she wishes within certain limitations (i.e., speed limits if the driver is law abiding, or certainly the technological limits of car performance). Only occasionally will each driver slow down because of the presence of other vehicles. As traffic increases to a more moderate level, encounters with slower moving vehicles are more numerous. It is still not difficult to pass slower-moving cars, and hence the drivers' average speed is not appreciably less than the desired speed. However, in heavy traffic changing lanes becomes difficult, and consequently the average speed of the traffic is lowered.

On the basis of these types of observations, we make a basic simplifying assumption that at any point along the road *the velocity of a car depends only on the density of cars,*

$$\boxed{u = u(\rho).} \qquad (61.2)$$

Lighthill and Whitham* and independently Richards† in the mid-1950s proposed this type of mathematical model of traffic flow.

If there are no other cars on the highway (corresponding to very low traffic densities), then the car would travel at the maximum speed $u_{\max}$,

$$\boxed{u(0) = u_{\max}.} \qquad (61.3)$$

$u_{\max}$ is sometimes referred to as the "mean free speed" corresponding to the velocity cars would move at if they were free from interference from other cars. However, as the density increases (that is as there are more and more cars per mile) eventually the presence of the other cars would slow the car down. As the density increases further the velocities of the cars would continue to diminish, and thus

$$\boxed{\frac{du}{d\rho} \equiv u'(\rho) \leq 0.} \qquad (61.4)$$

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*LIGHTHILL, M. J. and WHITHAM, G. B., "On kinematic waves II. A theory of traffic flow on long crowded roads," *Proc. Roy. Soc. A*, *229*, 317–345 (1955).

†RICHARDS, P. I., "Shock waves on the highway," *Operations Research 4*, 42–51 (1956).