MATH332-Linear Algebra	Homework Five Solutions

Eigenproblems, Diagonalization and Applications

Text: 5.1-5.3, 5.5 Section Overviews: 5.1-5.3, 5.5

Quote of Homework Five Solutions	
Limber Up	
	Zombieland : Rule #18 (2009)

1. Eigenvalues and Eigenvectors

$$\mathbf{A}_1 = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}, \quad \mathbf{A}_3 = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}, \quad \mathbf{A}_4 = \begin{bmatrix} .1 & .6 \\ .9 & .4 \end{bmatrix}, \quad \mathbf{A}_5 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix},$$

1.1. **Eigenproblems.** Find all eigenvalues and eigenvectors of $\mathbf{A}_i$ for $i = 1, 2, 3, 4, 5$.

Recall that the associated eigenproblem for a square matrix $\mathbf{A}_{n\times n}$ is defined by $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$ whose solution is found via the following auxiliary equations:

- Characteristic Polynomial : $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$
- Associated Null-space : $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$

For each of the previous matrices we have:

$$\begin{aligned} \det(\mathbf{A}_1 - \lambda\mathbf{I}) &= (4-\lambda)(1-\lambda)^2 + 2(1-\lambda) \\ &= (1-\lambda)(\lambda^2 - 5\lambda + 6) = 0 \implies \lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3 \end{aligned}$$

Case $\lambda_1 = 1$:

$$\left[\begin{array}{c|c} \mathbf{A}_1 - \lambda_1\mathbf{I} & \mathbf{0} \end{array}\right] = \left[\begin{array}{ccc|c} 3 & 0 & 1 & 0 \\ -2 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 3 & 0 & 1 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} 3x_1 = -x_3 \\ -2x_1 = 0 \\ x_2 \in \mathbb{R} \end{array} \Rightarrow \mathbf{x} = \begin{bmatrix} 0 \\ x_2 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

A basis for this eigenspace associated with $\lambda = 1$ is $B_{\lambda=1} = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

Case $\lambda_2 = 2$:

$$\left[\begin{array}{c|c} \mathbf{A}_1 - \lambda_2\mathbf{I} & \mathbf{0} \end{array}\right] = \left[\begin{array}{ccc|c} 2 & 0 & 1 & 0 \\ -2 & -1 & 0 & 0 \\ -2 & 0 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & 0 & 1 & 0 \\ -2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim$$

$$\sim \left[\begin{array}{ccc|c} 2 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_1 = -x_3/2 \\ x_2 = x_3 \\ x_3 \in \mathbb{R} \end{array} \Rightarrow \mathbf{x} = \begin{bmatrix} -1/2 \\ 1 \\ 1 \end{bmatrix} x_3$$

A basis for this eigenspace is $B_{\lambda=2} = \left\{ \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} \right\}$

Case $\lambda_3 = 3$:

$$\left[\begin{array}{c|c} \mathbf{A}_1 - \lambda_3 \mathbf{I} & \mathbf{0} \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ -2 & -2 & 0 & 0 \\ -2 & 0 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow$$

$$\Rightarrow \begin{array}{l} x_1 = -x_3 \\ x_2 = x_3 \\ x_3 \in \mathbb{R} \end{array} \Rightarrow \mathbf{x} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} x_3$$

A basis for this eigenspace is $B_{\lambda=3} = \left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

$$\det(\mathbf{A}_2 - \lambda \mathbf{I}) = (3-\lambda)(1-\lambda) - (-2) = \lambda^2 - 4\lambda + 5 \implies \lambda = \frac{-(-4) \pm \sqrt{16 - 4(1)(5)}}{2} = 2 \pm i$$

Case $\lambda = 2 \pm i$:

$$\left[\begin{array}{c|c} \mathbf{A}_2 - \lambda \mathbf{I} & \mathbf{0} \end{array}\right] = \left[\begin{array}{cc|c} 3 - (2 \pm i) & 1 & 0 \\ -2 & 1 - (2 \pm i) & 0 \end{array}\right] =$$

$$= \left[\begin{array}{cc|c} 1 \mp i & 1 & 0 \\ -2 & -1 \mp i & 0 \end{array}\right].$$

Row-reduction with complex numbers is possible. However, it is easier to note that for a two-by-two system we can use either row, in this case the first,

$$(1 \mp i)x_1 + 1x_2 = 0 \iff (1 \mp i)x_1 = -x_2, \tag{1}$$

to define the ratio between x_1 and x_2.[1] That is, if $x_1 = -1$ then $x_2 = 1 \mp i$ and thus the eigenvectors, like the eigenvalues, come in complex conjugate pairs $\mathbf{x} = [-1 \;\; 1 \mp i]^{\mathrm{T}}$.[2]

Since $\mathbf{A}_3$ is triangular we know the eigenvalues of$\mathbf{A}_3$ are,

$$\begin{array}{rcll} \lambda_1 & = & 4 & \text{(With algebraic multiplicity of 2)}, \\ \lambda_2 & = & 2 & \text{(With algebraic multiplicity of 2)}. \end{array}$$

Case $\lambda_1 = 4$:

$$\left[\begin{array}{c|c} \mathbf{A}_3 - \lambda_1 \mathbf{I} & \mathbf{0} \end{array}\right] = \left[\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 1 & 0 & 0 & -2 & 0 \end{array}\right] \Rightarrow \begin{array}{l} -2x_3 = 0 \\ x_1 = 2x_4 \\ x_2, x_4 \in \mathbb{R} \end{array} \Rightarrow \mathbf{x} = \begin{bmatrix} 2x_4 \\ x_2 \\ 0 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Thus the basis for this eigenspace is $B_{\lambda=4} = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

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[1]This only works for two-by-two problems. In higher dimensions it is not possible to fix one variable and uniquely define the remaining variables.

[2]For real matrices, complex Eigenvalues and eigenvectors must occur in conjugate pairs.

Case $\lambda = 2$:

$$\left[\begin{array}{c|c} \mathbf{A}_3 - \lambda_2 \mathbf{I} & \mathbf{0} \end{array}\right] = \left[\begin{array}{cccc|c} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_1 = 0 \\ x_2 = 0 \\ x_3, x_4 \in \mathbb{R} \end{array} \Rightarrow \mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

A basis for this eigenspace is $B_{\lambda=2} = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

$$\begin{aligned} \det\left(\begin{bmatrix} .1 - \lambda & .6 \\ .9 & .4 - \lambda \end{bmatrix}\right) &= (.4 - \lambda)(.1 - \lambda) - .54 = \lambda^2 - .5\lambda - .54 + .04 = \\ &= \lambda^2 - .5\lambda - .5 \Rightarrow \lambda = \frac{-(-.5) \pm \sqrt{(-.5)^2 - 4(1)(-.5)}}{2(1)} = \frac{.5 \pm 1.5}{2} \Rightarrow \lambda_1 = 1,\ \lambda_2 = -.5 \end{aligned}$$

Case $\lambda_1 = 1$:

(2) $$[\mathbf{A}_4 - \lambda_1 \mathbf{I} | \mathbf{0}] = \left[\begin{array}{cc|c} -.9 & .6 & 0 \\ .9 & .6 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} -.9 & .6 & 0 \\ 0 & 0 & 0 \end{array}\right] \Longrightarrow \mathbf{x}_1 = \begin{bmatrix} 2/5 \\ 3/5 \end{bmatrix}$$

Case $\lambda_2 = -.5$:

(3) $$[\mathbf{A}_4 - \lambda_2 \mathbf{I} | \mathbf{0}] = \left[\begin{array}{cc|c} .6 & .6 & 0 \\ .9 & .9 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} .6 & .6 & 0 \\ 0 & 0 & 0 \end{array}\right] \Longrightarrow \mathbf{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

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(4) $$\det(\mathbf{A}_5 - \lambda \mathbf{I}) = \lambda^2 - 1 = 0 \Longrightarrow \lambda = \pm 1$$

Case $\lambda = \pm 1$:

(5) $$[\mathbf{A}_5 - \lambda \mathbf{I} | \mathbf{0}] = \left[\begin{array}{cc|c} \mp 1 & -i & 0 \\ i & \mp 1 & 0 \end{array}\right] \Longrightarrow \mp x_1 - i x_2 = 0 \Longleftrightarrow \mp x_1 = i x_2 \Longrightarrow \mathbf{x} = \begin{bmatrix} i \\ \mp 1 \end{bmatrix}$$

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2. Applications of Diagonalization

2.1. **Eigenbasis and Decoupled Linear Systems.** Find the diagonal matrix $\mathbf{D}_i$ and vector $\tilde{\mathbf{Y}}$ that completely decouples the system of linear differential equations $\frac{d\mathbf{Y}_i}{dt} = \mathbf{A}_i \mathbf{Y}_i$ for $i = 3, 4, 5$.

If one finds $n-$many eigenvectors for an $n \times n$ matrix then it is possible to find a diagonal matrix *similar* to $\mathbf{A}_{n\times n}$. That is, if $\mathbf{A}$ has $n-$many eigenvectors then $\mathbf{A}$ has the following diagonal decomposition,

(6) $$\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1},$$

where $\mathbf{D}$ is a diagonal matrix whose elements are eigenvalues of $\mathbf{A}$ and $\mathbf{P}$ is an invertible matrix whose columns are the eigenvectors corresponding to the eigenvalue elements of $\mathbf{D}$. This sort of decomposition is important because if we recall the coordinate changes described in 02.LS.Geometry in $\mathbb{R}^n$ then we can see that the eigenvector matrix defines a coordinate change for a given linear problem,

(7) $$\frac{d\mathbf{Y}_i}{dt} = \mathbf{A}_i \mathbf{Y}_i = \mathbf{P}_i \mathbf{D}_i \mathbf{P}_i^{-1} \mathbf{Y}_i \Longleftrightarrow \frac{d\mathbf{P}_i^{-1}\mathbf{Y}_i}{dt} = \mathbf{D}_i \mathbf{P}_i^{-1} \mathbf{Y}_i \Longrightarrow \frac{d\tilde{\mathbf{Y}}_i}{dt} = \mathbf{D}_i \tilde{\mathbf{Y}}_i,$$

where $\tilde{\mathbf{Y}}_i = \mathbf{P}_i^{-1}\mathbf{Y}_i$ for $i = 1, 2, 3, 4, 5$. We then say that $\tilde{\mathbf{Y}}_i$ is the coordinates of $\mathbf{Y}_i$ relative to the eigenvector basis. What is interesting is that the problem, under this coordinate system, has become,

$$\frac{d\tilde{\mathbf{Y}}_i}{dt} = \mathbf{D}_i\tilde{\mathbf{Y}}_i \iff \frac{d}{dt}\begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \\ \tilde{y}_3 \\ \vdots \\ \tilde{y}_n \end{bmatrix} = \begin{bmatrix} d_{11} & 0 & 0 & \cdots & 0 \\ 0 & d_{22} & 0 & \cdots & 0 \\ 0 & 0 & d_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d_{nn} \end{bmatrix}\begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \\ \tilde{y}_3 \\ \vdots \\ \tilde{y}_n \end{bmatrix}, \tag{8}$$

which is completely decoupled and therefore solvable without any row-reduction or eigen-methods. For each of the systems $i = 3, 4, 5$ we have the following:

$$[\mathbf{P}_3|\mathbf{I}] \sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1/2 & 0 & 0 & 1 \end{array}\right] = \left[\mathbf{I}|\mathbf{P}_3^{-1}\right] \implies \tilde{\mathbf{Y}} = \mathbf{P}_3^{-1}\mathbf{Y} = \begin{bmatrix} y_2 \\ .5y_1 \\ y_3 \\ .5y_1 + y_4 \end{bmatrix}, \text{ and } \mathbf{D}_3 = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \tag{9}$$

$$\mathbf{P}_4 = \begin{bmatrix} 2/5 & 1 \\ 3/5 & -1 \end{bmatrix} \implies \mathbf{P}_4^{-1} = \begin{bmatrix} 1 & 1 \\ 3/5 & -2/5 \end{bmatrix}, \text{ and } \mathbf{D}_4 = \begin{bmatrix} 1 & 0 \\ 0 & -1/2 \end{bmatrix} \tag{10}$$

$$\mathbf{P}_5 = \begin{bmatrix} i & i \\ -1 & 1 \end{bmatrix} \implies \mathbf{P}_5^{-1} = \frac{1}{2i}\begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} = \frac{i}{2}\begin{bmatrix} -1 & i \\ -1 & -i \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -i & -1 \\ -i & 1 \end{bmatrix}, \text{ and } \mathbf{D}_5 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{11}$$

2.2. **Regular Stochastic Matrices.** For the *regular stochastic matrix* $\mathbf{A}_4$, define its associated steady-state vector, $\mathbf{q}$, to be such that $\mathbf{A}_4\mathbf{q} = \mathbf{q}$. Show that $\mathbf{q} = [2/5\ 3/5]^{\mathrm{T}}$.

2.3. **Limits of Time Series.** Show that $\lim\limits_{n\to\infty} \mathbf{A}_4^n\mathbf{x} = \mathbf{q}$ where $\mathbf{x} \in \mathbb{R}^2$ such that $x_1 + x_2 = 1$.
First, we note that we have already found the steady-state vector $\mathbf{q}$ since it is the eigenvector associated with $\lambda_1 = 1$. Now, the question is how to raise a matrix to an infinite power. Generally, it is unclear whether this processes converges and if it does, what it converges to. However, diagonalization offers us hope since,

$$\lim_{n\to\infty} \mathbf{A}^n = \lim_{n\to\infty} \mathbf{P}\mathbf{D}^n\mathbf{P}^{-1} = \mathbf{P}\lim_{n\to\infty} \mathbf{D}^n\mathbf{P}^{-1}, \tag{12}$$

where $[\mathbf{D}^n]_{ij} = d_{ii}^2\delta_{ij}$. Though calculating $\mathbf{A}^n$ is hard, calculating $\mathbf{D}^n$ is easy and more importantly, limiting processes on matrices now reduce to limiting processes on scalars, which is well-understood. In this case we have,

$$\lim_{n\to\infty} \mathbf{A}_4^n\mathbf{x} = \lim_{n\to\infty} \mathbf{P}_4\mathbf{D}_4^n\mathbf{P}_4^{-1}\mathbf{x} = \begin{bmatrix} 2/5 & 1 \\ 3/5 & -1 \end{bmatrix}\begin{bmatrix} \lim\limits_{n\to\infty} 1^n & 0 \\ 0 & \lim\limits_{n\to\infty}(-.5)^n \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 3/5 & -2/5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{13}$$

$$= \begin{bmatrix} .4 & .4 \\ .6 & .6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} .4(x_1 + x_2) \\ .6(x_1 + x_2) \end{bmatrix} = \begin{bmatrix} .4 \\ .6 \end{bmatrix} = \begin{bmatrix} 2/5 \\ 3/5 \end{bmatrix} = \mathbf{q}. \tag{14}$$

3. Theoretical Results

3.1. **Spectrum of Self-Adjoint Matrices.** Show that if $\mathbf{A} = \mathbf{A}^{\mathrm{H}}$ then $\sigma(\mathbf{A}) \subset \mathbb{R}$.

Proof: Define $q = \bar{\mathbf{x}}^{\mathrm{T}}\mathbf{A}\mathbf{x}$ and note that $q \in \mathbb{C}$. We want to show that $\bar{q} = q$, which impiles that $q \in \mathbb{R}$.

$$\bar{q} = {}^{-\mathrm{T}}\overline{\mathbf{A}\mathbf{x}} \tag{15}$$

$$= \bar{\bar{\mathbf{x}}}^{\mathrm{T}}\bar{\mathbf{A}}\bar{\mathbf{x}} \tag{16}$$

$$= \mathbf{x}^{\mathrm{T}}\bar{\mathbf{A}}\bar{\mathbf{x}} \tag{17}$$

$$= (\bar{\mathbf{x}}^{\mathrm{T}}\bar{\mathbf{A}}^{\mathrm{T}}\mathbf{x})^{\mathrm{T}} \tag{18}$$

$$= (\bar{\mathbf{x}}^{\mathrm{T}}\mathbf{A}^{\mathrm{H}}\mathbf{x})^{\mathrm{T}} \tag{19}$$

$$= (\bar{\mathbf{x}}^{\mathrm{T}}\mathbf{A}\mathbf{x})^{\mathrm{T}} \tag{20}$$

$$= q^{\mathrm{T}} = q \tag{21}$$

Hence, q is real. Now suppose that $\mathbf{x}$ is an eigenvector of $\mathbf{A}$ to get,

$$q = \bar{\mathbf{x}}^{\mathrm{T}}\mathbf{A}\mathbf{x} \tag{22}$$

$$= \bar{\mathbf{x}}^{\mathrm{T}}\lambda\mathbf{x} \tag{23}$$

$$= \lambda\bar{\mathbf{x}}^{\mathrm{T}}\mathbf{x} \tag{24}$$

$$= \lambda\sum_{i=1}^{n}\bar{x}_i x_i, \tag{25}$$

but $\bar{x}_i x_i = (\alpha_i - i\beta_i)(\alpha_i + i\beta_i) = \alpha^2 + \beta^2 \in \mathbb{R}$ implies that the summation is real. Since q is real the summation multiplied onto λ must be real. Thus, λ is real, which completes the proof.

3.2. **Connection to Invertible Matrices.** Show that if $\mathbf{A}$ is both diagonalizable and invertible then so is $\mathbf{A}^{-1}$.

If $\mathbf{A}$ is invertible then $\lambda \neq 0$, thus $\mathbf{D}$ associated with $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$ has a pivot in every column and is therefore invertible. Thus, $\mathbf{A}^{-1} = \mathbf{P}\mathbf{D}^{-1}\mathbf{P}^{-1}$ exists.

3.3. **Connection to Transposition.** Show that if $\mathbf{A}$ has $n-$many linearly independent eigenvectors then so does $\mathbf{A}^{\mathrm{T}}$.

If $\mathbf{A}$ has $n-$many linearly independent eigenvectors then $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$ and by transposition we have $\mathbf{A}^{\mathrm{T}} = (\mathbf{P}^{-1})^{\mathrm{T}}\mathbf{D}^{\mathrm{T}}\mathbf{P}^{\mathrm{T}} = \mathbf{Q}\mathbf{D}\mathbf{Q}^{-1}$ where $\mathbf{Q} = (\mathbf{P}^{-1})^{\mathrm{T}}$. Thus $\mathbf{A}^{\mathrm{T}}$ has a diagonalization, which impiles it has $n-$many linearly independent eigenvectors.

4. A Taste of Things to Come

Of the previous matrices only one of them has a direct relation to its Hermitian-adjoint. Recall, that in homework 2 we found that $\mathbf{A}_5^{\mathrm{H}} = \bar{\mathbf{A}}_5^{\mathrm{T}} = \mathbf{A}_5$ and that we called a matrix with this property self-adjoint. The following set of problems shows exactly how nice self-adjoint matrices are.

4.1. **Dot-Products Redux.** Recall that if $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$ then we define their dot-product as $\mathbf{x}\cdot\mathbf{y} = x_1y_1 + x_2y_2 \in \mathbb{R}$ and that this gave us some information about the angle between the two vectors. This formula is the same as the matrix-product $\mathbf{x}\cdot\mathbf{y} = \mathbf{x}^{\mathrm{T}}\mathbf{y}$ but if you apply it to the eigenvectors from $\mathbf{A}_5$ it will return non-sensible results. [3] The problem is that the vectors are not from $\mathbb{R}^2$ but are from $\mathbb{C}^2$. This problem is resolved by the Hermitian-adjoint. That is, whenever vectors are from $\mathbb{C}^2$ the dot-product is defined by, $\mathbf{x}\cdot\mathbf{y} = \mathbf{x}^{\mathrm{H}}\mathbf{y}$, which returns sensible results.[4] [5] With that said, show that the eigenvectors for $\mathbf{A}_5$ are orthogonal.

Vectors are orthogonal if their inner-product is zero. With our previous definition of inner-product, the calculation,

$$\mathbf{x}_{\mp}^{\mathrm{H}}\mathbf{x}_{\pm} = [\bar{i}\ \ \mp 1]\begin{bmatrix} i \\ \mp 1 \end{bmatrix} = [-i\ \ \pm 1]\begin{bmatrix} i \\ \mp 1 \end{bmatrix} = -i\cdot i + \mp 1\cdot\pm 1 = 1 - 1 = 0, \tag{26}$$

shows that the eigenvectors are orthogonal.

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[3]If you want to, try it! Consider $\mathbf{x}_1\cdot\mathbf{x}_1 = \mathbf{x}_1^{\mathrm{T}}\mathbf{x}_1 = 0$ seems to imply that the vector has no length. This is a problem.

[4]From this you will find that $\mathbf{x}_1\cdot\mathbf{x}_1 = \mathbf{x}_1^{\mathrm{H}}\mathbf{x}_1 = 2$, which implies that the length of the vector is $\sqrt{2}$. This makes more sense since we can think of this vector as pointing one-unit in both the real and imaginary directions, which creates a $1, 1, \sqrt{2}$ triangle.

[5]This might seem crazy but it is comforting to note that if the vectors are real then $\mathbf{x}^{\mathrm{H}} = \mathbf{x}^{\mathrm{T}}$ so this is merely and abstraction for complex number systems.

4.2. **Orthonormality.** Using this definition of dot-product scale both eigenvectors to be unit length.

An orthonormal basis is an orthogonal basis where the basis vectors have all been scaled to have unit-length. Using our definition of inner-product to define a length we note,

$$\sqrt{\mathbf{x}_{\mp}^{\mathrm{H}}\mathbf{x}_{\mp}} = \sqrt{1+1} = \sqrt{2}, \tag{27}$$

which implies that the normalized eigenvectors are, $\mathbf{x}_{\mp} = [i\sqrt{2}/2 \quad \mp\sqrt{2}/2]^{\mathrm{T}}$.

4.3. **Unitarity.** Using the normalized eigenvectors construct the matrix $\mathbf{U} = [\hat{\mathbf{x}}_1\ \hat{\mathbf{x}}_2]$ and show that $\mathbf{U}^{\mathrm{H}}\mathbf{U} = \mathbf{I}$.

Recall that $[\mathbf{U}^{\mathrm{H}}\mathbf{U}]_{ij} = \bar{\hat{\mathbf{x}}}\hat{\mathbf{x}}_j$ for $i = 1, 2$ and $j = 1, 2$, thus by the previous inner-products we have the identity matrix.

4.4. **Orthogonal Diagonalization.** It will later be shown that if a matrix is self-adjoint then it always provides enough eigenvectors for diagonalization and that these vectors can be chosen to be orthonormal. Moreover, a square matrix with orthonormal columns is called unitary and will always satisfy the property from the previous subsection. This effectively removes the need for inverse computations. Show this for our special matrix by verifying that $\mathbf{A}_5 = \mathbf{U}\mathbf{D}\mathbf{U}^{\mathrm{H}}$.

We have seen from the previous problems that if you have enough eigenvectors then it is possible to find a diagonal decomposition for the matrix. Geometrically, this decomposition provides a natural coordinate system for which the solution to the associated linear problem is manifestly clear. This is a powerful result but it can be made stronger.

The general statement is,

- If a matrix is self-adjoint then it can always be diagonalized.[6] Moreover, eigenvectors associated with different eigenvalues are orthogonal to one another and the resulting matrix can be constructed to have the property $\mathbf{P}\mathbf{P}^{\mathrm{H}} = \mathbf{P}^{\mathrm{H}}\mathbf{P} = \mathbf{I}$.[7]

Since $\mathbf{A}_5$ is self-adjoint we an demonstrate this fact.

$$\mathbf{U} = \begin{bmatrix} i\frac{\sqrt{2}}{2} & i\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \Longrightarrow \mathbf{U}^{\mathrm{H}} = \begin{bmatrix} -i\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ -i\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \text{ and } \mathbf{U}\mathbf{U}^{\mathrm{H}} = \begin{bmatrix} \frac{1}{2}+\frac{1}{2} & 0 \\ 0 & \frac{1}{2}+\frac{1}{2} \end{bmatrix}. \tag{28}$$

Thus, $\mathbf{A}_5 = \mathbf{U}\mathbf{D}_5\mathbf{U}^{\mathrm{H}}$ where the decomposition has been found without using row-reduction to find an inverse matrix!

4.5. **Spectral Representation.** This sort of decomposition has many applications and one is the so-called spectral representation of self-adjoint matrices. Show that $\mathbf{A}_5 = \lambda_1\hat{\mathbf{x}}_1\hat{\mathbf{x}}_1^{\mathrm{H}} + \lambda_2\hat{\mathbf{x}}_2\hat{\mathbf{x}}_2^{\mathrm{H}}$.

The previous result is quite powerful and can be used to derive other decompositions of the matrix $\mathbf{A}_5$. One such decomposition is called the spectral decomposition, which speaks to the action of $\mathbf{A}_5$ as a transformation. Assuming the given decomposition we consider the transformation,

$$\mathbf{A}_5\mathbf{y} = \left(\lambda_1\mathbf{x}_1\mathbf{x}_1^{\mathrm{H}} + \lambda_2\mathbf{x}_2\mathbf{x}_2^{\mathrm{H}}\right)\mathbf{y} \tag{29}$$
$$= \lambda_1\mathbf{x}_1\mathbf{x}_1^{\mathrm{H}}\mathbf{y} + \lambda_2\mathbf{x}_2\mathbf{x}_2^{\mathrm{H}}\mathbf{y} \tag{30}$$
$$= \lambda_1\left\langle\mathbf{x}_1, \mathbf{y}\right\rangle\mathbf{x}_1 + \lambda_2\left\langle\mathbf{x}_2, \mathbf{y}\right\rangle\mathbf{x}_2 \tag{31}$$

[6]It can also be shown that its eigenvalues are always real. This is important to the theory of quantum mechanics where the eigenvalues are hypothetical measurements associated with a quantum system. It would be disconcerting if you stuck a thermometer into a quantum-turkey and it somehow read $3+2i$. Yikes!

[7]If the eigenvectors from a shared eigenspace are not orthogonal then it is possible to orthogonalize them by the *Gram-Schmidt* process.

which implies that $\mathbf{A}_5$ transforms the vector $\mathbf{y}$ by projecting this vector onto each eigenvector, rescaling it by a factor of λ_i and then linearly combines the results. To demonstrate this decomposition we calculate the following outer-product,

$$\mathbf{x}_{\mp}\mathbf{x}_{\mp}^{\mathrm{H}} = \begin{bmatrix} i\frac{\sqrt{2}}{2} \\ \mp\frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} \bar{i}\sqrt{2}/2 & \mp\sqrt{2}/2 \end{bmatrix} \tag{32}$$

$$= \begin{bmatrix} i\frac{\sqrt{2}}{2} \\ \mp\frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} -i\sqrt{2}/2 & \mp\sqrt{2}/2 \end{bmatrix} \tag{33}$$

$$= \begin{bmatrix} 1/2 & \mp i/2 \\ \pm i/2 & 1/2 \end{bmatrix}, \tag{34}$$

which gives,

$$\mathbf{A}_5 = 1 \cdot \begin{bmatrix} 1/2 & -i/2 \\ i/2 & 1/2 \end{bmatrix} - 1 \cdot \begin{bmatrix} 1/2 & i/2 \\ -i/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}. \tag{35}$$

5. Lecture Appreciation

In lecture we considered the following applications of linear algebra:

- Ridged geometric preserving transformations of $\mathbb{R}^n$ as it relates to ridged body and fluid dynamics.
- Normal mode analysis of vibrational systems.
- Numerical approximation of solutions to partial differential equations.
- General solutions to constant linear ordinary differential equations.
- Quantum bits, Lie structures and the vector-space $\mathbb{C}^2$.

Pick one of the topics and:

(1) In five paragraphs or less, summarize the lecture.
(2) Address as many questions raised in lecture as possible.
(3) List the remaining questions that you have.