Anisotropic media: birefringence.

in an <u>isotropic</u> medium:

$$\vec{D} = \varepsilon \vec{E}$$

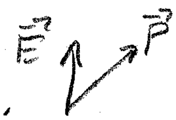
if linear, ε doesn't depend on $\vec{E}$

$\vec{D} \parallel \vec{E}$

in an anisotropic medium, $\vec{D} \nparallel \vec{E}$

$$\vec{D} = \underset{\substack{\uparrow \\ \text{applied}}}{\vec{E}} + 4\pi \underset{\substack{\uparrow \\ \text{induced polarization}}}{\vec{P}}$$

b/c of asymmetry of crystal, $\vec{P} \nparallel \vec{E}$ e.g.



write $\vec{D} = \overleftrightarrow{\varepsilon} \cdot \vec{E}$

$$\overleftrightarrow{\varepsilon} = \text{dielectric tensor} = \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{pmatrix}$$

from thermodynamics, $\overleftrightarrow{\varepsilon}$ is <u>symmetric</u> $\varepsilon_{ij} = \varepsilon_{ji}$

We can choose the orientation of coordinates to diagonalize

$$\rightarrow \begin{pmatrix} D_x \\ D_y \\ D_z \end{pmatrix} = \begin{pmatrix} \varepsilon_x & & \\ & \varepsilon_y & \\ & & \varepsilon_z \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

this coord. system is aligned along the crystal axes.

so if $\vec{E} = E_x(x,y,z)\,\hat{x}$,

$\vec{D} = D_x \hat{x} = \varepsilon_x E_x$ only.

biaxial crystal: $\varepsilon_x \neq \varepsilon_y \neq \varepsilon_z$

uniaxial crystal: one pair is the same. e.g. $\varepsilon_x = \varepsilon_y \neq \varepsilon_z$

isotropic crystal $\varepsilon_x = \varepsilon_y = \varepsilon_z = \varepsilon$ $\quad \vec{D} = \varepsilon \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \vec{E}$

Wave propagation in birefringent media.
(non magnetic, no free charge)

$$\nabla\cdot\vec{D}=0 \qquad \nabla\cdot\vec{H}=0$$

$$\nabla\times\vec{E}=-\frac{1}{c}\frac{\partial\vec{H}}{\partial t} \qquad \nabla\times H=\frac{1}{c}\frac{\partial\vec{D}}{\partial t}$$

all fields will share same time dependence $\sim e^{-i\omega t}$

$$\frac{\partial}{\partial t}\to -i\omega$$

there will be a plane wave solution $\sim e^{i\vec{k}\cdot\vec{r}}$
- don't know $\vec{k}$ but can solve for it

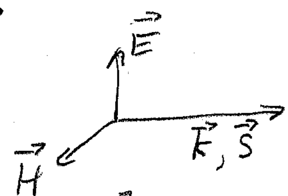
$$\vec{\nabla}\to i\vec{k}$$

$$\to \vec{k}\times\vec{E}=\frac{\omega}{c}\vec{H} \qquad k\times\vec{H}=-\frac{\omega}{c}\vec{D}$$

in an isotropic medium, $\vec{D}=\varepsilon\vec{E}$ and

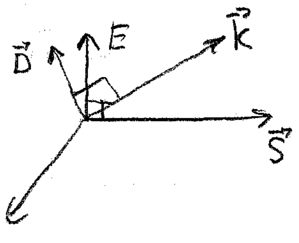
$\vec{H}\perp\vec{k},\vec{E}$ $\qquad$ $\vec{E}\perp\vec{k},\vec{H}$

$\therefore\ \vec{k}\perp\vec{E}\perp\vec{H}$



$\vec{S}=\frac{c}{4\pi}\vec{E}\times\vec{H}$ is in $\vec{k}$ direction

here $\vec{H}\perp\vec{k},\vec{E}$ still $\vec{D}\perp\vec{H}$ also, but $\vec{D}\nparallel\vec{E}$



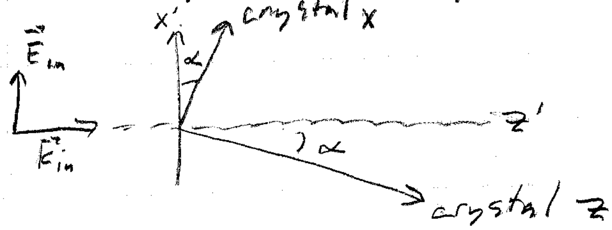
wavefronts are $\perp\vec{k}$

power flows along $\vec{S}$: like "ray" in geom. optics

Walkoff

consider plane wave normally incident on surface

- input $\vec{E}$ along $\vec{S}$ outside crystal
- input is at angle to crystal axes:



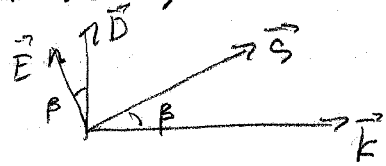
boundary conditions: $D^{\perp}$ and $E^{\parallel}$ are continuous.

know that $\vec{E} \nparallel \vec{D}$ $\therefore$ one of them bends.

$\vec{D}_{in}$ must be $\parallel$ to $\vec{D}_{out}$

but $\vec{E}_{out} = E_{in}\,\hat{x}' + E_{new}\,\hat{z}'$ is ok.

$\therefore$ inside crystal



beam can refract even at normal inc.

but opposite polarization does not refract if it's along crystal y axis

→ double image for unpolarized light

two rays walk off from eachother: angle β

$$\cos\beta = \frac{\vec{E}\cdot\vec{D}}{E_0 D_0} = \frac{D_x^2/\epsilon_x + D_y^2/\epsilon_y + D_z^2/\epsilon_z}{E_0 D_0}$$

$$\vec{D}_{in} = D_0\,\hat{x}' = D_0\cos\alpha\,\hat{x} - D_0\sin\alpha\,\hat{z} \qquad D_y = 0$$

$$\cos\beta = \frac{D_0^2\left(\cos^2\alpha/\epsilon_x + \sin^2\alpha/\epsilon_z\right)}{D_0^2\left(\frac{\cos^2\alpha}{\epsilon_x^2} + \frac{\sin^2\alpha}{\epsilon_z^2}\right)^{1/2}\underbrace{\left(\cos^2\alpha \cancel{+\sin^2\alpha}\right)^{1/2}}_{1}}$$

calcite: $n_x = 1.658$ $n_z = 1.486$ $\alpha = 35°$ → $\beta = 6.1°$

The index ellipsoid.

in many cases, we need to know the phase velocity of the wave.

let $\vec{k} = \frac{\omega}{c} \vec{n}$

normally, $\vec{n}$ is $\sqrt{\varepsilon}\, \hat{k}$ (isotropic)

phase velocity is $\omega / |k| = c / |n|$

if we know direction of k w.r.t. crystal axes $\rightarrow |n|$

work with uniaxial case. $\varepsilon_x = \varepsilon_y = \varepsilon_o = n_o^2$ "ordinary"

$\varepsilon_z = \varepsilon_e = n_e^2$ extraordinary axis

from Maxwell $\vec{H} = \vec{n} \times \vec{E}$, $D = -\vec{n} \times \vec{H} = -\vec{n} \times (\vec{n} \times \vec{E})$

now $\vec{D} = \vec{n} \times (\vec{E} \times \vec{n}) = n^2 \vec{E} - (\vec{n} \cdot \vec{E}) \vec{n}$

write in index form

$$D_i = \sum_k \varepsilon_{ik} E_k = n^2 E_i - \sum_k n_i n_k E_k$$

or
$$\sum_k \underbrace{(n^2 \delta_{ik} - n_i n_k - \varepsilon_{ik})}_{\text{matrix}} E_k = 0$$

set of 3 sim. eqns.

Find det |matrix| = 0

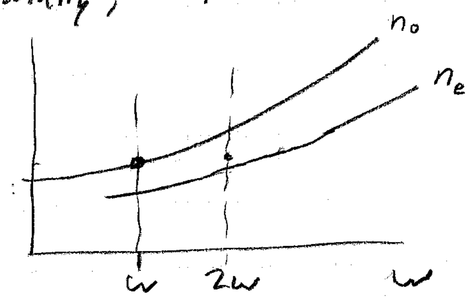
expand out $\rightarrow$ eqn 97.10 L+L

principal axes $\varepsilon^{(x)}, \varepsilon^{(y)}, \varepsilon^{(z)}$

$$n^2 \left(\varepsilon^{(x)} n_x^2 + \varepsilon^{(y)} n_y^2 + \varepsilon^{(z)} n_z^2 \right) - \left[n_x^2 \varepsilon^{(x)} (\varepsilon^{(y)} + \varepsilon^{(z)}) + n_y^2 \varepsilon^{(y)} (\varepsilon^{(x)} + \varepsilon^{(z)}) + n_z^2 \varepsilon^{(z)} (\varepsilon^{(x)} + \varepsilon^{(y)}) \right] + \varepsilon^{(x)} \varepsilon^{(y)} \varepsilon^{(z)} = 0$$

Application: phase-matching in second harmonic generation.
b/c of non-lin response at high inten;
can induce non-lin polarization $\rightarrow$ $\vec{E}_{2\omega} \perp \vec{E}_{\omega}$
in a birefringent medium.

to maximize efficiency, must have
$V_{ph}(2\omega) = V_{ph}(\omega)$
i.e. $n(2\omega) = n(\omega)$
but normally, n increases w/ ω



place ω along n_o
tune angle so
$n_o(\omega) = n_e(2\omega)$

this is "type I" phase matching.

uniaxial case $\varepsilon^{(x)} = \varepsilon^{(y)} = \varepsilon_o \qquad \varepsilon^{(z)} = \varepsilon_e$

$$n^2\left(\varepsilon_o(n_x^2+n_y^2) + \varepsilon_e n_z^2\right) - n_x^2\varepsilon_o(\varepsilon_o+\varepsilon_e) - n_y^2\varepsilon_o(\varepsilon_o+\varepsilon_e) - n_z^2\varepsilon_e\cdot 2\varepsilon_o$$
$$+ \varepsilon_o^2\varepsilon_e = 0$$

this factors to

$$(n^2-\varepsilon_o)\left(\varepsilon_e n_z^2 + \varepsilon_o(n_x^2+n_y^2) - \varepsilon_o\varepsilon_e\right) = 0$$

$\therefore$ 2 possible waves $\quad n^2 = \varepsilon_o \quad$ ordinary wave

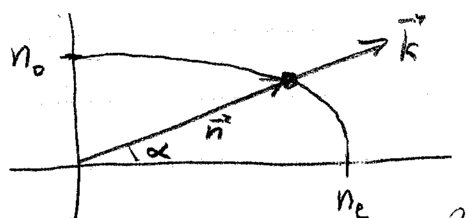
or $\quad \dfrac{n_x^2+n_y^2}{n_e^2} + \dfrac{n_z^2}{n_o^2} = 1 \quad$ extraordinary wave.

let α be angle btw $\vec{k}$ and crystal $\hat{z}$ (c-axis)

now we have a variable index for the e-wave:

$$n_e^2(\alpha)\left(\frac{\sin^2\alpha}{n_e^2} + \frac{\cos^2\alpha}{n_o^2}\right) = 1$$

Geometric construction



for a given $\vec{k}$, $\vec{D}$ can be oriented along 2 principal ways (polarization)

- o-wave is insensitive to α
- e-wave: α tunes $n_e(\alpha)$ from n_o to n_e.