Note Title 1/21/2008

1_21_08

chapter 2 in Griffiths

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V\Psi$$

usually V is assumed to be independent of time

Sep. of variables

$$\Psi(x,t) = \psi(x)\,\varphi(t)$$

$$\frac{\partial \Psi}{\partial t} = \psi(x)\,\dot{\varphi}(t)$$

$$\frac{\partial^2 \Psi}{\partial x^2} = \psi''(x)\,\varphi(t)$$

insert in and divide by $\psi\varphi$

$$\underbrace{i\hbar\frac{\dot{\varphi}}{\varphi}}_{f(t)} = \underbrace{-\frac{\hbar^2}{2m}\frac{\psi''}{\psi} + V(x)}_{g(x)}$$

must be constant

$[\hbar]$ = Energy * time [think $E=\hbar\omega$]

Hence so $\left[\hbar\frac{\dot{\phi}}{\varphi}\right]$ = Energy

$$i\hbar \frac{\dot{\varphi}}{\varphi} = -\frac{\hbar^2}{2m}\frac{\psi''}{\psi} + V(x) = E$$

$$i\hbar\dot{\varphi} - E\varphi = 0 \qquad 2 \quad \boxed{-\frac{\hbar^2}{2m}\psi'' + V\psi = E\psi}$$

$$1 \quad \boxed{\dot{\varphi} = -\frac{i}{\hbar}E\varphi}$$

$$1 \qquad \varphi(t) = C e^{-i/\hbar\, Et}$$

Recall from 311. To solve an initial, Boundary value problem for the string, we had to take a superposition of solutions of the time equation. This led to a Fourier Series.

Here we have the same issue. The solution to 1 is not general. But it is a theorem that:

Every solution to the time dependent Schrödinger Equation (TDSE) can be written as

$$\Psi(x,t) = \sum_{n=1}^{\infty} c_n \psi_n(x) e^{-iE_n t/\hbar}$$

where ψ_n is a solution to the spatial part with energy E_n:

A solution of the form $\psi(x) e^{-iEt/\hbar}$ is a stationary state. why

$$|\Psi(x,t)|^2 = \psi^*(x) e^{iEt/\hbar} \cdot \psi(x) e^{-iEt/\hbar}$$

$$= \psi^*(x)\psi(x) = |\psi(x)|^2$$ indep. of time

meanwhile

(2) $$\boxed{-\frac{\hbar^2}{2m}\psi'' + V\psi = E\psi}$$

$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi$ is the result of our <u>canonical</u> substitution

$p \to -i\hbar\frac{\partial}{\partial x}$ in the classical kinetic energy:

$$H = \frac{p^2}{2m} + V \Longrightarrow \hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V$$

So the spatial part of Schröd. is $\boxed{\hat{H}\psi = E\psi}$

Thus E is an ε-value of $\hat{H}$ with ε-vector ψ.

$$\langle H \rangle = \int \psi^* \underbrace{\hat{H}\psi}_{E\psi} \, dx = E\int \psi^*\psi \, dx$$

$$= E$$

similarly $\langle H^2 \rangle = \int \psi^* H^2 \psi \, dx$

$$= \int \psi^* H \, H \psi \, dx$$

$$= \int \psi^* H \, E \psi \, dx$$

$$= E \int \psi^* H \psi \, dx = E^2 \int \psi^* \psi \, dx = E^2$$

So $\sigma_H^2 = \langle H^2 \rangle - \langle H \rangle^2 = E^2 - E^2 = 0$

So for an eigenstate of H (i.e. a separable solution)

Every measurement of energy yields the value E.

Each different allowed value of E will have a different wave function

$$\Psi_1(x,t) = \psi_1(x) e^{-iE_1 t/\hbar}$$

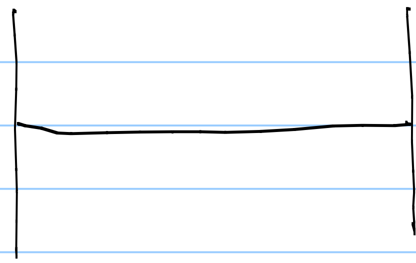
$$\Psi_2(x,t) = \psi_2(x) e^{-iE_2 t/\hbar}$$

$$\vdots$$

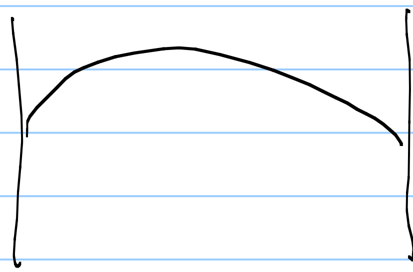
11/9/07
311 lecture notes

Classical analogy

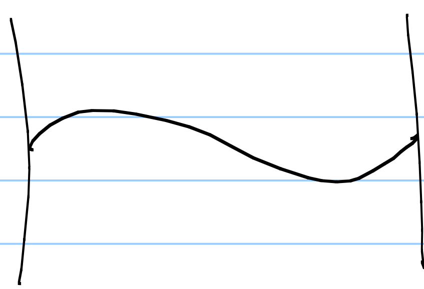
$\psi_N(x) = \sin(N\pi x/\ell)$ $\qquad$ $\omega_N = \frac{c}{\pi\ell}N$



$\omega_0 = 0$



$\omega_1 = c\pi/\ell$



$\omega_2 = \frac{2c\pi}{\ell}$

Suppose you're given some initial state $\psi(x,0)$

Strategy: solve the TISE for the (infinite) set of state
$\psi_1(x), \psi_2(x), \dots \psi_N(x)$

Then find the coefficients c_n such that

$$\Psi(x,0) = \sum_{n=1}^{\infty} c_n \psi_n(x)$$

then

$$\Psi(x,t) = \sum_{n=1}^{\infty} c_n \psi_n(x) e^{-iE_n t/\hbar}$$

$$= \sum_{n=1}^{\infty} c_n \Psi_n(x,t)$$

The separable solutions

$$\Psi_n(x,t) = \psi_n(x) e^{-iE_n t/\hbar}$$ are

stationary states

Example suppose

$$\Psi(x,0) = c_1 \psi_1(x) + c_2 \psi_2(x)$$

for simplicity assume $c_{1,2}$ & $\psi_{1,2}$ are real.

$$\Psi(x,t) = c_1 \psi_1(x) e^{-iE_1 t/\hbar} + c_2 \psi_2 e^{-iE_2 t/\hbar}$$

$$|\Psi|^2 = \Psi^* \Psi$$

$$= \left(c_1 \psi_1 e^{iE_1 t/\hbar} + c_2 \psi_2 e^{iE_2 t/\hbar}\right)$$

$$\times \left(c_1 \psi_1 e^{-iE_1 t/\hbar} + c_2 \psi_2 e^{-iE_2 t/\hbar}\right)$$

$$= c_1^2 \psi_1^2 + c_2^2 \psi_2^2 + c_1 c_2 \psi_1 \psi_2 e^{i(E_1 - E_2) t/\hbar} + c_1 c_2 \psi_1 \psi_2 e^{i(E_2 - E_1) t/\hbar}$$

$$= c_1^2 \psi_1^2 + c_2^2 \psi_2^2 + 2 c_1 c_2 \psi_1 \psi_2 \cos\left((E_2 - E_1) t/\hbar\right)$$

See mathematica animation

Consider the following mathemat. statement

$(\psi, H\psi)$ I will follow the c.c. convention on page 94.

$$= \int \psi^* H \psi$$

$$= \int \psi^* \frac{-\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2}$$

integrate by parts twice to move the $\frac{\partial^2}{\partial x^2}$ from ψ to ψ^*.

show that

$$= \int -\frac{\hbar^2}{2m} \frac{\partial^2 \psi^*}{\partial x^2} \psi \, dx$$

$$(H\psi, \psi)$$

if everything were real we would say that H is symmetric.

For complex inner products, we say H is Hermitian