# 3: Second harmonic wave generation and propagation

- Separation of wave eqn into frequency channels (Boyd 2.1)
- Derive equations for SHG
  - Boyd 2.2 (sum-freq version), 2.7 (first part)
- Non-depleted pump solution
  - Boyd 2.7
  - Perfect phase matching
  - Including phase mismatch

## Nonlinear wave equation

- Generalize for NL polarization

$$\nabla^2\mathbf{E}-\frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}=\mu_0\frac{\partial^2\mathbf{P}}{\partial t^2}$$

  - For now, neglect vector character of response
  - Expand polarization as a Taylor series
    - Any 1/n! factors are included in definition of χ's

$$P=\varepsilon_0\left(\chi^{(1)}E+\chi^{(2)}E^2+\chi^{(3)}E^3+\cdots\right)$$

  - Separate linear from NL part: $P=\varepsilon_0\chi^{(1)}E+P^{NL}$
  - Important: $\chi^{(1)}$ is evaluated at the ω of the E-field.
  - Now $P^{NL}$ is the source term to the linear eqn

$$\nabla^2\mathbf{E}-\frac{n^2}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}=\mu_0\frac{\partial^2\mathbf{P}^{NL}}{\partial t^2}$$

**This expression assumes that both sides are harmonic at the same ω**

# Signal channels

- We've seen that the nonlinear polarization can have many frequency components ($\omega_n$) and wave directions ($\mathbf{k}_n$)
- Total field is sum of all components:

$$\mathbf{E}(\mathbf{r},t)=\sum_{n>0}\mathbf{E}_n(\mathbf{r},t)=\sum_{n>0}\vec{\mathcal{E}}_n(\mathbf{r},t)\cos\left[\mathbf{k}_n\cdot\mathbf{r}-\omega_n t\right]$$

**Real field**

$$=\sum_{n>0}\left[\mathbf{A}_n(r,t)\exp\left(i\left(\mathbf{k}_n\cdot\mathbf{r}-\omega_n t\right)\right)+c.c.\right]$$

  - In general, there can be different k's at the same frequency $\omega_n$ (e.g. diffraction from NL grating)
  - With this convention, field envelopes are $\boxed{\mathbf{A}_n=\tfrac{1}{2}\vec{\mathcal{E}}_n}$
- Similarly, $\mathbf{P}(\mathbf{r},t)=\sum_{n>0}\left[\mathbf{P}_n(r,t)\exp\left(i\left(\mathbf{k}_n\cdot\mathbf{r}-\omega_n t\right)\right)+c.c.\right]$

# Signal channels: frequency separation

- Put sum of different harmonic components into WE

$$\left(\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\sum_{m>0}\left[\mathbf{A}_m(r,t)\exp\left(i\left(\mathbf{k}_m\cdot\mathbf{r}-\omega_m t\right)\right)+c.c.\right]$$

$$=\mu_0\varepsilon_0\frac{\partial^2}{\partial t^2}\sum_{m>0}\chi_m^{(1)}\left[\mathbf{A}_m(r,t)\exp\left(i\left(\mathbf{k}_m\cdot\mathbf{r}-\omega_m t\right)\right)+c.c.\right]$$

$$+\mu_0\frac{\partial^2}{\partial t^2}\sum_{m>0}\left[\mathbf{P}_m^{NL}(r,t)\exp\left(i\left(\mathbf{k}_m\cdot\mathbf{r}-\omega_m t\right)\right)+c.c.\right]$$

- Collect terms with same $\omega_m$

$$\left(\nabla^2-\frac{1}{c^2}\left(1+\chi_n^{(1)}\right)\frac{\partial^2}{\partial t^2}\right)\mathbf{A}_m(r,t)e^{i(\mathbf{k}_m\cdot\mathbf{r}-\omega_m t)}=\mu_0\frac{\partial^2}{\partial t^2}\mathbf{P}_m^{NL}(r,t)e^{i(\mathbf{k}_m\cdot\mathbf{r}-\omega_m t)}$$

$n_m^2$

## Intensity calculation

- Time average intensity can be calculated from the field:

$$I_n = \tfrac{1}{2}\varepsilon_0 nc\left|\vec{\mathcal{E}}_n\right|^2 = \tfrac{1}{2}\varepsilon_0 nc\,\vec{\mathcal{E}}_n\cdot\vec{\mathcal{E}}_n^{\,*}$$

- With the convention that $\boxed{\mathbf{A}_n = \tfrac{1}{2}\vec{\mathcal{E}}_n}$

$$I_n = 2\varepsilon_0 nc\left|\mathbf{A}_n\right|^2 = 2\varepsilon_0 nc\,\mathbf{A}_n\cdot\mathbf{A}_n^{*}$$

- Now we can write the field over a sum of ± frequencies

$$\mathbf{E}(\mathbf{r},t) = \sum_n\left[\mathbf{A}_n(r,t)\exp\left(i\left(\mathbf{k}_n\cdot\mathbf{r}-\omega_n t\right)\right)+c.c.\right]$$

## Eqns for second harmonic generation

- Start with frequency-separated inhomogeneous wave equation
  - For simplicity, assume all waves are CW plane waves, propagating in z-direction, polarized in x-direction
  - Assume A's depend on z only

$$\left(\frac{\partial^2}{\partial z^2}-\frac{n_m^{\;2}}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{A}_m(z)\mathrm{e}^{i(\mathbf{k}_m\cdot\mathbf{r}-\omega_m t)} = \mu_0\frac{\partial^2}{\partial t^2}\mathbf{P}_m(z)\mathrm{e}^{i(\mathbf{k}_m\cdot\mathbf{r}-\omega_m t)}$$

- Find NL polarization term that oscillates at the frequency $\omega_m$ that is on the RHS

## Eqns for second harmonic generation

- Evaluate NL polarization, suppress vector direction

$$P^{(2)} = \varepsilon_0 \chi^{(2)} \left( A_1 \mathrm{e}^{i(\mathbf{k}_1 \cdot \mathbf{r} - \omega_1 t)} + A_2 \mathrm{e}^{i(\mathbf{k}_2 \cdot \mathbf{r} - \omega_2 t)} + c.c. \right)^2$$

- Pick out terms with the same time dependence as RHS
  - At $\omega_2 = 2\omega_1$, $P^{(2)}(2\omega_1) = \varepsilon_0 \chi^{(2)} {A_1}^2 \mathrm{e}^{i2(\mathbf{k}_1 \cdot \mathbf{r} - \omega_1 t)}$

$$\left( \frac{\partial^2}{\partial z^2} - \frac{{n_2}^2}{c^2} \frac{\partial^2}{\partial t^2} \right) A_2(z) \mathrm{e}^{i(\mathbf{k}_2 \cdot \mathbf{r} - \omega_2 t)} = \mu_0 \varepsilon_0 \chi^{(2)} \frac{\partial^2}{\partial t^2} A_1(z) \mathrm{e}^{i2(\mathbf{k}_1 \cdot \mathbf{r} - \omega_1 t)}$$

  - Evaluate time derivatives, cancel common time dependence

$$\frac{\partial^2}{\partial z^2} \left( A_2(z) e^{i k_2 z} \right) + \frac{\omega_2^2 n_2^2}{c^2} A_2(z) e^{i k_2 z} = -\frac{\omega_2^2}{c^2} \chi^{(2)} {A_1}^2(z) e^{i 2 k_1 z}$$

## Equation for SH amplitude growth

- Evaluate z derivatives

$$\left( \frac{\partial^2 A_2}{\partial z^2} + 2 i k_2 \frac{\partial A_2}{\partial z} - k_2^2 A_2 \right) e^{i k_2 z} + \frac{\omega_2^2 n_2^2}{c^2} A_2 e^{i k_2 z} = -\frac{\omega_2^2}{c^2} \chi^{(2)} {A_1}^2 e^{i 2 k_1 z}$$

- Terms in blue cancel, divide out $k_2 z$ term

$$\frac{\partial^2 A_2}{\partial z^2} + 2 i k_2 \frac{\partial A_2}{\partial z} = -\frac{\omega_2^2}{c^2} \chi^{(2)} {A_1}^2 e^{i(2 k_1 z - k_2 z)}$$

- Compare terms on RHS:
  - scale length for growth of amplitude ~ L

$$\frac{\partial^2 A_2}{\partial z^2} \sim \frac{1}{L^2} A_2, \quad 2 i k_2 \frac{\partial A_2}{\partial z} \sim \frac{4\pi}{L \lambda_2} A_2$$

## Slowly-varying amplitude approx (SVEA)

- If changes in amplitude are slower than λ, i.e. $L \gg \frac{\lambda}{4\pi}$
- then we can drop the second derivative from the equation:

$$2ik_2 \frac{\partial A_2}{\partial z} = -\frac{\omega_2^2}{c^2}\chi^{(2)} A_1{}^2 e^{i(2k_1 z - k_2 z)} \rightarrow \frac{\partial A_2}{\partial z} = i\frac{\omega_2^2}{2k_2 c^2}\chi^{(2)} A_1{}^2 e^{i(2k_1 z - k_2 z)}$$

- Since eqn was 2nd order, this means second solution (counter-propagating wave) is ignored. That wave would in general interfere with the forward wave, leading to interference on a λ scale.

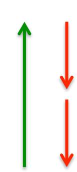
## Second equation for fundamental

- Look for a 2nd equation for the wave oscillating at $\omega_1$
- Calculate NL polarization at $\omega_1$
- Get resulting wave equation using SVEA

$$\left(\frac{\partial^2}{\partial z^2} - \frac{n_2{}^2}{c^2}\frac{\partial^2}{\partial t^2}\right) A_1 e^{i(k_1 z - \omega_1 t)} = \mu_0 \varepsilon_0 \chi^{(2)} \frac{\partial^2}{\partial t^2} 2 A_2 A_1{}^* e^{i(k_2 z - \omega_2 t) - i(k_1 z - \omega_1 t)}$$

$$\frac{\partial A_1}{\partial z} = i\frac{\omega_1^2}{k_1 c^2}\chi^{(2)} A_2 A_1{}^* e^{-i(2k_1 z - k_2 z)}$$

- Notes
  - factor of 2 on RHS because of two combinations
  - $A_2$: unconjugated: up arrow
  - $A_1$*: down arrow
  - This eqn leads to depletion of fund, back conversion

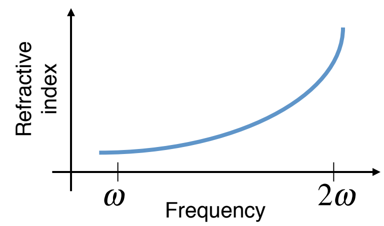


## non-depleted pump: SH solution

- If we assume $A_1$ is not depleted, $A_1$ = constant
- Solve only one equation

$$\frac{\partial A_2}{\partial z} = i\frac{\omega_2^2}{2k_2c^2}\chi^{(2)}A_1^{\,2}e^{i(2k_1z-k_2z)}$$

- Phase mismatch results from n




$$\Delta k \equiv 2k_1 - k_2 = 2\frac{\omega_1}{c}(n_1 - n_2)$$

- Assuming phase mismatch = 0 (somehow)

$$A_2(z) = i\frac{\omega_2^2}{2k_2c^2}\chi^{(2)}A_1^{\,2}z$$

  - ***Field*** grows linearly in z

## SH intensity: non-depleted pump

- Calculate intensity, using our convention for amplitude

$$I_n = 2\varepsilon_0 nc\left|\mathbf{A}_n\right|^2 = 2\varepsilon_0 nc\,\mathbf{A}_n\cdot\mathbf{A}_n^{\,*}$$

$$I_2(z) = 2\varepsilon_0 n_2 c\left|A_2\right|^2 = 2\varepsilon_0 n_2 c\left(\frac{\omega_2^2\chi^{(2)}}{2k_2c^2}\right)^2\left|A_1\right|^4 z^2$$

$$= \frac{1}{2\varepsilon_0 n_2 c}\left(\frac{\omega_2^2\chi^{(2)}}{2k_2c^2}\right)^2 I_1^{\,2}z^2$$

## SHG without phase-matching

Non-depleted pump approximation: treat $A_1$ as constant

$$\frac{\partial A_2}{\partial z} = i\frac{\omega_2{}^2 d}{k_2 c^2} A_1{}^2 e^{i\Delta k z} \qquad d \equiv \chi^{(2)}/2 \qquad \text{Integrate:} \quad A_2(L) = i\frac{\omega_2{}^2 d}{k_2 c^2} A_1{}^2 \int_0^L e^{i\Delta k z}\, dz$$

$$A_2(L) = i\frac{\omega_2{}^2 d}{k_2 c^2} A_1{}^2 L \frac{\left(e^{i\Delta k L} - 1\right)}{i\Delta k L}$$

Convert to intensity $I_2 = 2\varepsilon_0 n_2 c |A_2|^2$

$$\rightarrow \frac{1}{2\varepsilon_0 n_2 c} I_2(z) = \left(\frac{1}{2\varepsilon_0 n_1 c}\right)^2 I_1{}^2 \left(\frac{\omega_2 d}{n_2 c}\right)^2 L^2 \left(\frac{\sin(\Delta k L/2)}{\Delta k L/2}\right)^2$$

$$\rightarrow I_2(L) = \frac{\omega_2{}^2 d^2}{2\varepsilon_0 n_1{}^2 n_2 c^3} I_1{}^2 L^2 \,\text{sinc}^2(\Delta k L/2)$$

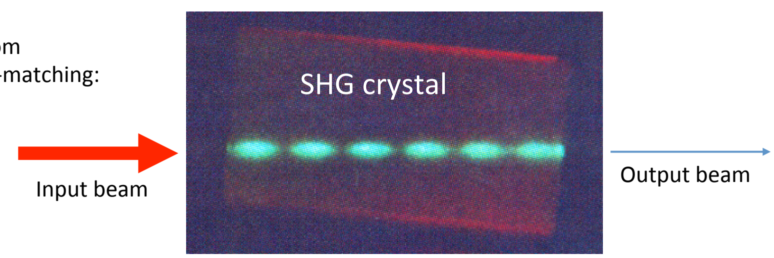
As a function of L and fixed |Δk|>0: $I_2(L) = \frac{\omega_2{}^2 d^2}{2\varepsilon_0 n_1{}^2 n_2 c^3} I_1{}^2 \frac{4}{\Delta k^2} \sin^2(\Delta k L/2)$

Yield oscillates:

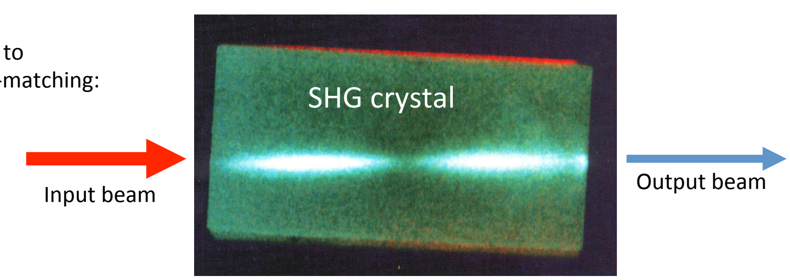
- Period = "coherence length" $L_{coh} = 2\pi/\Delta k$
- Amplitude proportional to $\max(I_2) \propto 1/\Delta k^2$

## Light created in real crystals

Far from phase-matching:




Closer to phase-matching:




Note that SH beam is brighter as phase-matching is achieved.