MACS 332 - June 29, 2006
Exam I - 50 minutes

NAME: Key

In order to receive full credit, SHOW ALL YOUR WORK. Full credit will be given only if all reasoning and work is provided. Please enclose your final answers in boxes.

1. (10 points)
Given the following linear system,

$$3x_1 - 9x_2 + 6x_3 = 0 \quad (1)$$
$$-x_1 + 3x_2 - 2x_3 = 0. \quad (2)$$

a. Determine the general solution set.
b. What geometric object does the general solution set correspond to?

The linear system (1),(2) corresponds to the augmented matrix,

a. $\left[\begin{array}{ccc|c} 3 & -9 & 6 & 0 \\ -1 & 3 & -2 & 0 \end{array}\right] \overset{R2 = 3R2 + R1}{\sim} \left[\begin{array}{ccc|c} 3 & -9 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \overset{R1 = R1/3}{\sim} \left[\begin{array}{ccc|c} 1 & -3 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

which implies that

$x_1 = 3x_2 - 2x_3$
x_2 is free
x_3 is free

$\Rightarrow \vec{x} = \begin{bmatrix} 3x_2 - 2x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \quad (*)$

is the general solution.

b. Vector Equation (*) corresponds to a plane in $\mathbb{R}^3$.

2. (10 points) Conceptual questions. Briefly respond to the following statements.

- Systems of linear equations have three possible solution sets.
  a. Describe all three.
  b. Give a geometric example for each one.

The three possible solution sets are,

1. No Solutions
2. ∞-many solutions
3. 1-unique solution

In $\mathbb{R}^2$: 2 lines can

1. Parallel Lines which are not colinear
2. Colinear lines
3. 2 lines intersecting at a point.

In $\mathbb{R}^3$, planes can. See pg 102 problem 3 for pictures.

1. Intersect Each other but not at a common set
   Be parallel with Each other.
2. Intersect at a line.
   Be the same plane.
3. Intersect at a point.

- Suppose none of the vectors in the set $S = \{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$, $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$, is a multiple of one of the other vectors. Is S a linearly independent set? Justify your response.

If a set is linearly independent then no vector from the set can be made from a linear combination of other vectors in the set.

Though none of the vectors can be constructed by a constant multiple of the others, this does not imply that it cannot be a linear combination of the others.

Thus, it is unknown whether the set S is linearly independent or not.

3. (10 points) Do the following vectors span $\mathbb{R}^3$. Justify your answer.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 0 & -2 & 4 \\ 1 & -1 & 1 & 3 \end{bmatrix} \underset{\sim}{\overset{R2=R2-2R1,\ R3=R3-R1}{}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & -4 & 2 \\ 0 & -2 & 0 & 2 \end{bmatrix} \overset{R3=R3-R2}{\sim} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & -4 & 2 \\ 0 & 0 & 4 & 0 \end{bmatrix}$$

Since there is a pivot for Each Row we have that the columns of

$A = [v_1\ v_2\ v_3\ v_4]$ span $\mathbb{R}^3$

by theorem 1.4.

4. (10 points) Assuming that A is non-singular, prove that $(\mathbf{A}^{\mathrm{T}})^{-1} = (\mathbf{A}^{-1})^{\mathrm{T}}$.

$(A^T)^{-1}(A^T) = I$

$(A^T)^{-1}(A^T) = (A^{-1})^T(A^T) = (AA^{-1})^T = (I)^T = I.$

$\boxtimes$

5. (10 points) Find the inverse of the following matrix **and** check your result with an appropriate matrix product.

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ -3 & 1 & 4 & 0 & 1 & 0 \\ 2 & -3 & 4 & 0 & 0 & \end{array}\right] \underset{R3=R3-2R1}{\overset{R2=R2+3R1}{\sim}} \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & -3 & 8 & -2 & 0 & 1 \end{array}\right] \sim$$

$$\overset{R3=R3+3R2}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & 0 & 2 & 7 & 3 & 1 \end{array}\right] \overset{\substack{R1=R1+R3 \\ R2=R2+R3}}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 8 & 3 & 1 \\ 0 & 1 & 0 & 10 & 4 & 1 \\ 0 & 0 & 2 & 7 & 3 & 1 \end{array}\right] \sim$$

$$\overset{R3=R3/2}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 8 & 3 & 1 \\ 0 & 1 & 0 & 10 & 4 & 1 \\ 0 & 0 & 1 & 7/2 & 3/2 & 1/2 \end{array}\right]$$

$$AA^{-1} = \begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix} \begin{bmatrix} 8 & 3 & 1 \\ 10 & 4 & 1 \\ 7/2 & 3/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 8-7 & 3-3 & 1-1 \\ -24+10+14 & -9+4+6 & -3+1+2 \\ 16-30+14 & 6-12+6 & 2-3+2 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \checkmark$$