PHGN 462 Homework 12

1) As you may be aware, the point of particle accelerators is to accelerate charged particles. Lately we've learned that accelerating particles radiate. Sometimes this is a good thing: We can make a radiation source for imaging or medical treatments simply by accelerating charges appropriately. But sometimes it's a bad thing. Atom smashers try to get particles up to very high energies before doing the smashy, and if the particles are constantly losing energy to radiation, that doesn't really help any.

Particle accelerators come in linear and circular varieties, so let's check both of those out and see how radiation losses work in each geometry.

a) For a linear accelerator, start from Lienard's formula and show that the ratio of (power radiated by the accelerated particle) to (power supplied to the accelerated particle) is given by:

$$\frac{P_{rad}}{P_{sup}} = \frac{q^2}{6\pi\varepsilon_0 m^2 c^3}\frac{1}{v}\frac{dU}{dx}$$

Where U is the energy of the particle. Note that relativistically we still have $F = \frac{dp}{dt}$, as long as we use the relativistic momentum, $p = \gamma m v$. Recall also that $F = \frac{dU}{dx}$ and that a power looks like $\frac{dU}{dt}$.

Two identities that you might need (I needed them) are:

$$\frac{d\gamma}{dt} = \gamma^3 \frac{va}{c^3} \qquad\qquad \frac{dp}{dt} = m\gamma^3 a$$

Where v and a are the particle's speed and acceleration, and γ is the Lorentz factor from special relativity. If you use one or both of these, prove that they are true. The proofs should be short, so if yours is getting long, you may be going off-track.

b) It looks like the size of the loss compared to the size of the gain depends on the applied force $\frac{dU}{dx}$. The harder we push are charged particles, the worse the losses are. But are the losses meaningful in any situation we're likely to encounter? Suppose were in a highly relativistic regime, where $v \approx c$. Calculate what $\frac{dU}{dx}$ needs to be for the ratio $\frac{P_{rad}}{P_{sup}}$ to be close to one, first for an electron, and then for a proton. Express your answer in eV/m. Comment on and make sense of what you find.

c) The results of (b) hopefully demonstrated that linear accelerators are very efficient: Radiative losses just aren't that significant in any scenario that we're going to be able to reach anytime soon. The only drawback linear accelerators have is that in order to reach particle energies on the scale we're shooting for these days in high energy physics, the accelerators would need to be hundreds of thousands of kilometers long. This is, of course, a *fairly significant* drawback. Thus, we turn to circular accelerators so that we can accelerate particles over arbitrary distances.

Circular accelerators introduce a new feature: In addition to applying linear acceleration to the particles (to speed them up), we have to apply centripetal acceleration to push them in a circle. As it turns out, the radiative losses from the centripetal acceleration can be pretty bad. Since we know from (b) that linear acceleration losses are negligible, henceforth we'll ignore tangential accelerations and consider our particles to be undergoing uniform circular motion so that we can get a clear look at the centripetal effects.

Again assuming highly relativistic circumstances, show that for a circular accelerator the power lost per distance traveled is given by:

$$\frac{dU}{dx} = \frac{q^2 U^4}{6\pi\varepsilon_0 m^4 c^8 R^2}$$

This is, in effect, the extra force we'd need to apply just to overcome the centripetal portion of the radiation losses.

d) Assuming a particle energy of 50 GeV and an accelerator radius of 167 m, punch in numbers and find the energy loss per meter for an electron and for a proton. Express your answer in eV/m (or MeV/m or GeV/m as appropriate). Comment on and make sense of what you find.

2) (Read *all* of the below)

This problem is based on Pollack and Stump 15.3, and does a decent job of putting a nice, happy bow on all of classical electromagnetism.

Starting from Maxwell's equations with sources $\rho(\vec{x},t)$ and $\vec{J}(\vec{x},t)$, show that:

$$-\nabla^2\vec{E} + \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} = -\frac{1}{\varepsilon_0}\nabla\rho - \mu_0\frac{\partial\vec{J}}{\partial t}$$

$$-\nabla^2\vec{B} + \frac{1}{c^2}\frac{\partial^2\vec{B}}{\partial t^2} = \mu_0\nabla\times\vec{J}$$

Once you've done that, let's expand on things a bit. Maxwell's equations in differential form,

$$\nabla\cdot\vec{E} = \frac{\rho}{\varepsilon_0} \qquad \nabla\cdot\vec{B} = 0$$

$$\nabla\times\vec{E} = \frac{-\partial\vec{B}}{\partial t} \qquad \nabla\times\vec{B} = \mu_0\vec{J} + \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}$$

sometimes cause people to conclude odd things because they treat changing electric and magnetic fields as sources on an equal footing with charges and currents. But of course, those aren't *really* equal. Ultimately the changing electric and magnetic fields themselves came from charges and currents. Indeed, eventually all E and B fields come from charges and currents, period. The equations that this problem had you derive make that clearer: They're differential equations for E & B that explicitly lay out charges and currents as the ultimate sources of the fields.

If you ever take graduate E&M, you may have the good fortune to solve these differential equations using Green's functions to get the following integral solutions:

$$\vec{E}(x,t) = \frac{1}{4\pi\varepsilon_0}\int\left\{\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}\rho(\vec{x}',t') + \frac{\vec{x}-\vec{x}'}{c|\vec{x}-\vec{x}'|^2}\frac{\partial\rho(\vec{x}',t')}{\partial t'} - \frac{1}{c^2|\vec{x}-\vec{x}'|}\frac{\partial\vec{J}(\vec{x}',t')}{\partial t'}\right\}d^3x'$$

$$\vec{B}(x,t) = \frac{\mu_0}{4\pi}\int\left\{\vec{J}(\vec{x}',t')\times\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3} + \frac{\partial\vec{J}(\vec{x}',t')}{\partial t'}\times\frac{\vec{x}-\vec{x}'}{c|\vec{x}-\vec{x}'|^2}\right\}d^3x'$$

with everything in the integrand evaluated at the retarded time, as usual. These are known as Jefimenko's equations, and are the complete integral prescription for figuring out fields, given complete information about the charges and currents in the neighborhood. Hopefully you're still reading, because this is the place where I tell you that part of the credit for this problem comes from you commenting on how awesome this Jefimenko perspective is. Also examine the structure of the Jefimenko equations and comment on whether they make any intuitive sense and why (or why not, if that's how you feel).