EXERCISES

63.1. (a) Briefly explain the four basic properties of a velocity-density relationship:
(1) $u(\rho_{max}) = 0$
(2) $u(0) = u_{max}$
(3) $du/d\rho \leq 0$
(4) $dq/d\rho$ decreases as ρ increases (i.e., $d^2q/d\rho^2 < 0$).

(b) Assume that the velocity depends on the density in a linear way, $u(\rho) = \alpha + \beta\rho$. Show that in order for this to satisfy the properties in part (a), $\alpha = u_{max}$ and $\beta = -u_{max}/\rho_{max}$.

(c) What is the flow as a function of density? Sketch the Fundamental Diagram of Road Traffic.

(d) At what density is the flow maximum? What is the corresponding velocity? What is the maximum flow?

63.2. If cars obey state laws on following distances (refer to exercise 61.1), what is a road's capacity if the speed limit is 50 m.p.h. (80 k.p.h.)? At what density and velocity does this maximum flow occur? Will increasing the speed limit increase the road's capacity?

63.3. Some traffic data was compared to a flow-density relationship of the following from:

$$q(\rho) = \rho(\alpha - \beta\rho).$$

The best fit (in a least-squares sense) occurred for

$$\alpha = 58.6 \text{ miles/hr}; \quad \beta = .465 \text{ (miles)}^2\text{/hr}.$$

(a) What is the maximum density?
(b) What is the maximum velocity?
(c) What is the maximum flow?
(d) Guess what type of road this is.

63.4. Using the Lincoln Tunnel data of Sec. 62, sketch the flow as a function of density (the Fundamental Diagram of Road Traffic).

63.5. Show that if $u''(\rho) \leq 0$ (and $u'(\rho) \leq 0$), then the flow-density curve has only one local maximum.

63.6. Assume that $u = u(\rho)$. If α equals a car's acceleration, show that

$$\alpha = -\rho \frac{du}{d\rho} \frac{\partial u}{\partial x}.$$

Is the minus sign reasonable?

63.7. Consider exercise 61.3. Suppose that drivers accelerate such that

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{a^2}{\rho}\frac{\partial \rho}{\partial x},$$

where a is a positive constant.

(a) Physically interpret this equation.
(b) If u only depends on ρ and the equation of conservation of cars is valid, show that

$$\frac{du}{d\rho} = -\frac{a}{\rho}.$$

(c) Solve the differential equation in part (b), subject to the condition that $u(\rho_{max}) = 0$. The resulting flow-density curve fits quite well to the Lincoln Tunnel data.
(d) Show that a is the velocity which corresponds to the road's capacity.
(e) Discuss objections to this theory for small densities.

63.8. If $u = u_{max}(1 - \rho^3/\rho_{max}^3)$, what is the capacity of the road?

64. Steady-State Car-Following Models

In this section we will suggest a method for determining the specific observed velocity-density relationships. In order to explain the velocity-density curve, we can carefully analyze actual drivers making their driving decisions. The mathematical model suggested here is motivated by both its being reasonable and by the results of some experiments run for the purposes of developing such a model.

Consider the nth car on a highway, $x_n(t)$. As before, we assume cars cannot pass each other (frequently a rather rash assumption). We postulate that an individual car's motion only depends on the car ahead. Theories of this sort are called **car-following** models. In the simplest such model, we assume that a car's acceleration is proportional to the relative velocity,

$$\frac{d^2x_n(t)}{dt^2} = -\lambda\left(\frac{dx_n(t)}{dt} - \frac{dx_{n-1}(t)}{dt}\right). \qquad (64.1)$$

If the car following is going faster than the preceding one, then the car following will slow down (and thus $\lambda > 0$). The larger the relative velocity, the more the car behind accelerates or decelerates. λ measures the sensitivity of the two-car interaction. However, equation 64.1 suggests that acceleration or deceleration occurs instantaneously. Instead, let us allow some time delay before the driver reacts to changes in the relative velocity. This process is modeled by specifying the acceleration at a slightly later time,

$$\frac{d^2x_n(t+T)}{dt^2} = -\lambda\left(\frac{dx_n(t)}{dt} - \frac{dx_{n-1}(t)}{dt}\right), \qquad (64.2)$$

where T is the reaction time. Mathematically, this equation represents a sys-