Note Title 9/28/2007

If $A \cdot \vec{x}_0 = 0 = \vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix}$ N times

$A\vec{x} = \vec{y}$

$A(\vec{x} + \vec{x}_0) = A \cdot \vec{x} + \underbrace{A \cdot \vec{x}_0}_{0}$

$= A \cdot \vec{x} = \vec{y}$

$A \in R^{N \times M}$

$A \quad \begin{bmatrix} a_{11} & a_{12} & & a_{1m} \\ a_{21} & & & \vdots \\ a_{31} & & & | \\ a_{N1} & - & - - & a_{Nm} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

9-28-07

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From Last time

$A \in R^{N \times m}$

the null space consists of all vectors x such that

$A \in R^{N \times M}$ $A \cdot \vec{x} = 0$

$\downarrow$ must be in R^M

$\hookrightarrow$ really $= \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in R^N$

what is a (Linear) space?

A set of vectors closed under addition of vectors and multiplication by scalars

Let $N(A)$ denote all vectors in $\mathbb{R}^m$ such that

$$A \cdot \vec{x} = 0$$

Then suppose we have 2 elements of $N(A)$, say $\vec{z}_1$ and $\vec{z}_2$. we must show that

$$\vec{z}_1 + \vec{z}_2 \in N(A) \text{ and}$$

$$\alpha \vec{z}_1 \in N(A) \quad \text{for}$$

any $\vec{z}_1, \vec{z}_2, \alpha$.

$$A(\vec{z}_1 + \vec{z}_2) = \underbrace{A \cdot \vec{z}_1}_{0} + \underbrace{A \cdot \vec{z}_2}_{0}$$

by assump.

hence $z_1 + z_2 \in N(A)$.

$$A(\alpha \vec{z}_1) = \alpha A \cdot \vec{z}_1 = 0 \quad \checkmark$$

$N(A)$ is a space

Since all the vectors in $N(A)$ are also in $\mathbb{R}^m$

this makes $N(A)$ a subspace of $\mathbb{R}^m$.

One more space

Let $A \in \mathbb{R}^{n \times m}$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & & a_{2m} \\ \vdots & & & & \\ a_{n1} & & \cdots & & a_{nm} \end{bmatrix}$$

Let $\vec{a}_1$ be the first column of A. $\vec{a}_j$ the jth column.

The span$\{\vec{a}_1, \vec{a}_2, \vec{a}_3, \ldots \vec{a}_m\}$

is denoted $R(A)$ (for "range")

is this a space? Yes

Since $R(A)$ consists of all Linear comb of the vectors this set is clearly closed under addition & scalar mult.

Each col. of A has n elements. so

$$\underbrace{R(A) \subset}_{\text{subset}} R^n$$

$$N(A) \subset R^m$$

Example $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$\det(A) = -1 - 1 = -2 \neq 0$

Hence A^{-1} exists

$A^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2} A$

normalize the columns of A

$$\left.\begin{aligned} \vec{a}_1 &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ \vec{a}_2 &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{aligned}\right\} \quad \begin{aligned} \|\vec{a}_1\|^2 &= 1 \\ \|a_2\|^2 &= 1 \end{aligned}$$

$\tilde{A} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$\tilde{A} = \tilde{A}^{-1}$

$\tilde{A} = \tilde{A}^T$ } symmetric matrix property

if $A \in R^{n \times m}$ then

$A^T \in R^{m \times n}$ $A^T A$ $A A^T$

$(A^T)_{ij} = A_{ji}$

Def. of transpose

Now consider the two vectors

$\vec{a}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\vec{a}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

notice that $\vec{a}_1 \cdot \vec{a}_2 = \frac{1}{2}(1-1) = 0$

and $\vec{a}_2 \cdot \vec{a}_1 = \vec{a}_1 \cdot \vec{a}_2 = 0$

So the columns of A are orthogonal (perpendicular)

Since they are normalized we call them

orthonormal

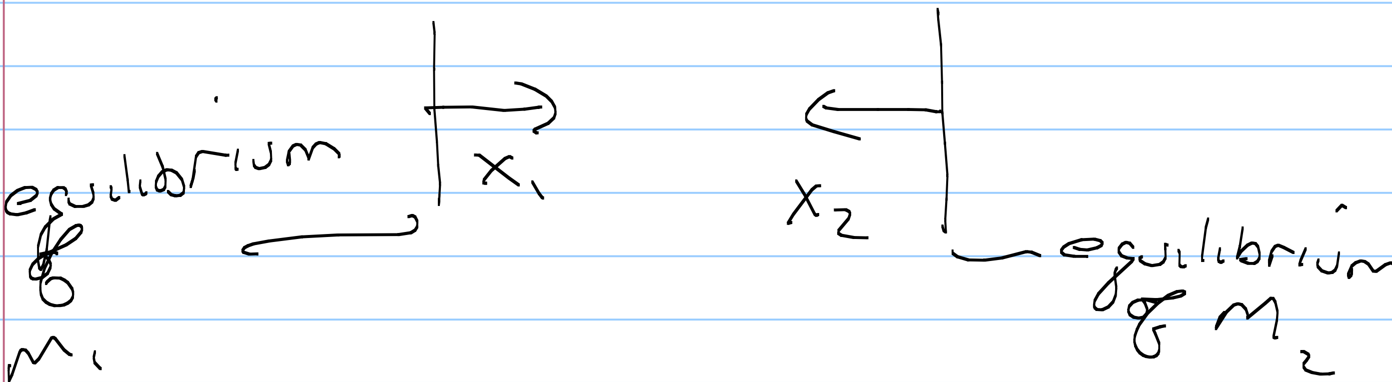
orthogonal   normalized

$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ orthogonal matrix

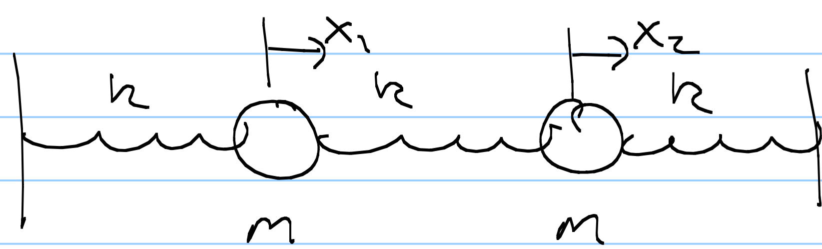
$Q^T Q = Q Q^T = I$

Example: coupled springs/masses

F=ma for the coupled masses



Keep it simple $m_1 = m_2$ $k_1 = k_2 = k_3$



F=ma $\Rightarrow$ $m\ddot{x}_1 + kx_1 + k(x_1 - x_2) = 0$

$$m\ddot{x}_2 + kx_2 + k(x_2 - x_1) = 0$$

in matrix form

$$\underbrace{\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}}_{M} \underbrace{\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix}}_{\ddot{\vec{u}}} + \underbrace{\begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix}}_{\equiv K} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\vec{u}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\boxed{M\ddot{\vec{u}} + K\vec{u} = 0}$$

or $$\boxed{\ddot{\vec{u}} + M^{-1}K\vec{u} = 0}$$

is M always invertible

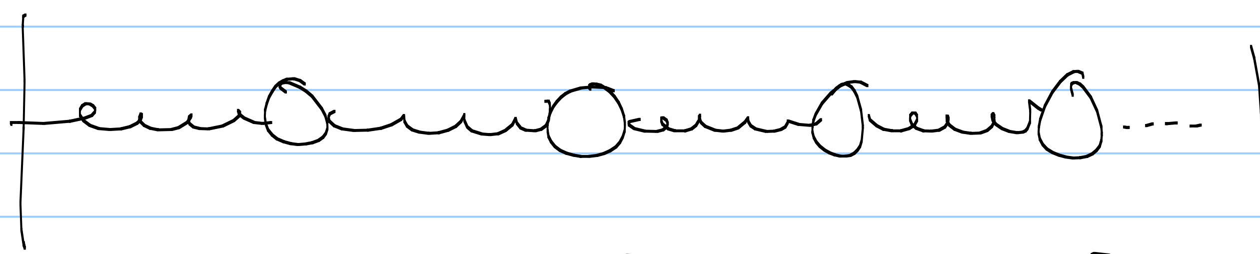
$$\mathrm{Det}(M) = m_1 m_2$$

$$M^{-1}K = \begin{bmatrix} \frac{2k}{m} & -\frac{k}{m} \\ -\frac{k}{m} & \frac{2k}{m} \end{bmatrix} = \omega_0^2 \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$= \omega_0^2 \left\{ \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

General Case

monoatomic Lattice



$$\frac{1}{\omega_0^2}\ddot{\vec{u}} = \frac{1}{\omega_0^2}\begin{bmatrix} \ddot{x}_1 \\ \vdots \\ \ddot{x}_N \end{bmatrix} = \begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ 0 & 1 & -2 & 1 & \\ 0 & 0 & 1 & -2 & 1 \\ & & & \ddots & \\ & & 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}$$

$$\boxed{\frac{1}{\omega_0^2}\ddot{\vec{u}} = K\vec{u}}$$

can show that in the limit that $N \to \infty$ $\vec{x} \Rightarrow f(x)$

$$K \to \frac{\partial^2}{\partial x^2}$$

becomes the wave quation!

$$A\vec{x} \qquad A \in R^{N \times N} \quad x \in R^N$$

$$A\vec{x} = \lambda \vec{x}$$

$$= \lambda I_{N \times N} \vec{x}$$

$$\Rightarrow \underbrace{(A - \lambda I)}_{\text{matrix}} \vec{x} = 0$$

matrix $\quad \vec{x} \Rightarrow 0$

must be that $\vec{x} \in$

$$N(A - \lambda I) \Rightarrow \text{Det}(A - \lambda I) = 0$$

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\text{Det}\left[\begin{pmatrix} 1-\lambda & 1 \\ 1 & -1-\lambda \end{pmatrix} \right] = 0$$

$$(1-\lambda)(-1-\lambda)-1=0$$

$$+(1-\lambda)(1+\lambda)=-1$$

$$1-\lambda^2=-1$$

$$2=\lambda^2 \Rightarrow \lambda=\pm\sqrt{2}$$

$$A\begin{pmatrix}x\\y\end{pmatrix}=\pm\sqrt{2}\begin{pmatrix}x\\y\end{pmatrix}$$

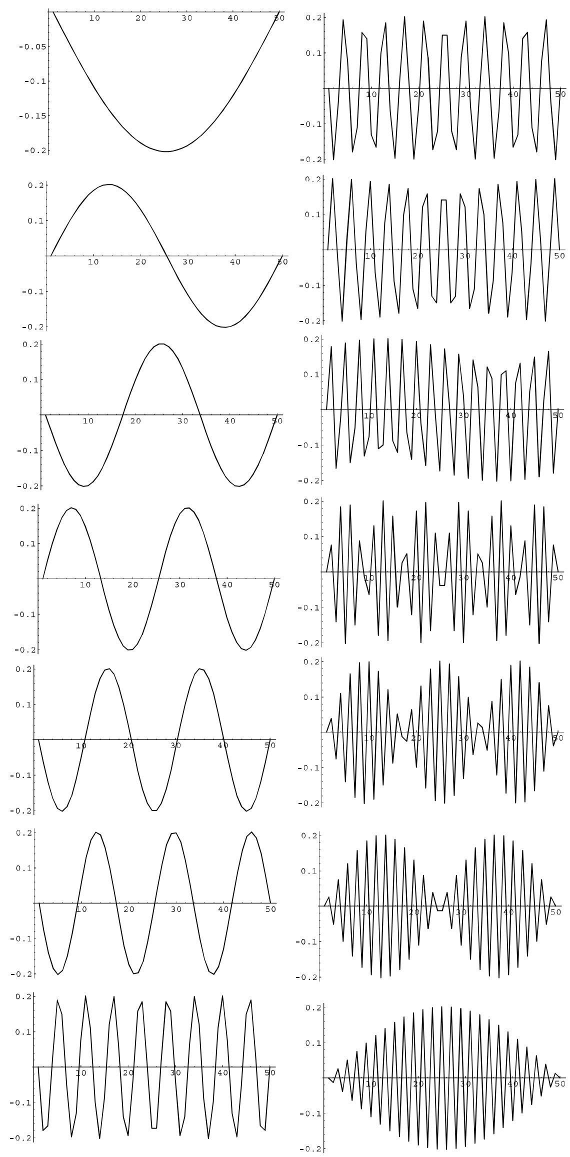


Figure 1.12: A sample of the normal modes (free oscillations) of a homogeneous 50 point lattice with fixed ends. The lower frequency modes are purely sinusoidal; the higher frequency modes become modulated sinusoids as a result of the dispersive effects of this being a discrete system.

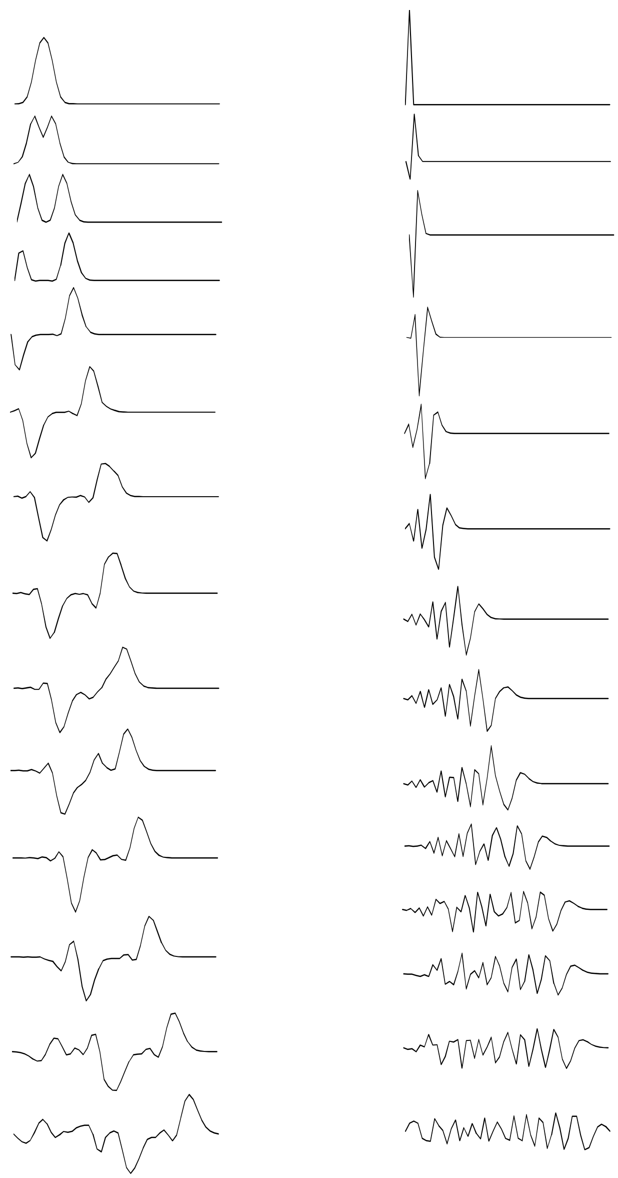


Figure 1.13: Waves on a lattice (discrete string). The two columns of figures are snapshots in time of waves propagating in a 1D medium. The only difference between the two is the initial conditions, shown at the top of each column. On the right, we are trying to propagate an inpulsive function (a displacement that is turned on only for one grid point in space and time). On the left, we are propagating a smoothed version of this. The medium is homogeneous except for an isolated reflecting layer in the middle. The dispersion seen in the right side simulation is the result of the discretness of the medium: waves whose wavelengths are comparable to the grid spacing sense the granularity of the medium and therefore propagate at a slightly different speed than longer wavelength disturbances.