

E. Kreyszig, Advanced Engineering Mathematics, 9th ed. Section 8.3, pgs. 345-349

Lecture: Special Matrices Module: 08

Suggested Problem Set: Suggested Problems : {5, 7, 10, 17}

June 25, 2009

E. Kreyszig, Advanced Engineering Mathematics, 9th ed. Section 8.4, pgs. 349-356

Lecture: Diagonalization Module: 08

Suggested Problem Set: Suggested Problems : {4, 7, 12, 15, 17}

June 25, 2009

E. Kreyszig, Advanced Engineering Mathematics, 9th ed. Section 8.5, pgs. 3356-363

Lecture: Complex Matrices Module: 08

Suggested Problem Set: Suggested Problems : {5, 6, 9, 12, 16}

June 25, 2009

Quote of Lecture 8	
When I get to the bottom I go back to the top of the slide. Where I stop and turn and I go for a ride 'till I get to the bottom and I see you again.	
	The Beatles : The White Album (1968)

At this point we have the system $\mathbf{Ax} = \lambda\mathbf{x}$, which is an eigenvalue/eigenvector problem for the square matrix \mathbf{A} . We solve this problem by finding roots of the characteristic equation $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ and then defining a basis for the null-space of $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$ we find the eigenvectors of \mathbf{A} . Now, we consider what can be done with this information but before this we present two statements important to this development:

1. We say that two matrices are **similar** if there exists \mathbf{P} such that $\mathbf{A} = \mathbf{PBP}^{-1}$ and we call the previous equality a **similarity transformation** of the matrix \mathbf{A} . This implies two things:

(a) $\det(\mathbf{A} - \lambda\mathbf{I}) = \det(\mathbf{B} - \lambda\mathbf{I})$ ¹

(b) $\sigma(\mathbf{A}) = \sigma(\mathbf{B})$

2. Diagonalization Theorem: Let \mathbf{A} be a square-matrix such that the algebraic multiplicity is equal to the geometric multiplicity for each eigenvalue.² Then there exists an invertible matrix \mathbf{P} and diagonal matrix \mathbf{D} such that,

$$\mathbf{A} = \mathbf{PDP}^{-1},$$

where \mathbf{P} is a matrix whose columns are the eigenvectors of \mathbf{A} and \mathbf{D} is a diagonal matrix whose elements are the eigenvalues of \mathbf{A} corresponding to the 'eigen-columns' of \mathbf{P} .³ Notes:

- We call this particular similarity transformation the **diagonalization** of \mathbf{A} .
- The conclusion is that if we have enough eigenvectors then the eigenbasis can be used as a coordinate system for the column space of the matrix and under this coordinate system the matrix is diagonal and is populated with its own spectrum.

¹To see this we consider the determinant of $\mathbf{B} - \lambda\mathbf{I}$ to find $\det(\mathbf{B} - \lambda\mathbf{I}) = \det(\mathbf{P}^{-1}(\mathbf{A} - \lambda\mathbf{I})\mathbf{P}) = \det(\mathbf{A} - \lambda\mathbf{I})$.

²This implies that the $\mathbf{A}_{n \times n}$ gives rise to n -many eigenvectors. If this were not the case then we wouldn't know how to populate the \mathbf{P} matrix.

³It can be shown that eigenvectors are naturally linearly independent, which implies that the columns of \mathbf{P} form a linearly independent set and thus \mathbf{P} is an invertible matrix.

- If it is also true that if \mathbf{A} is self-adjoint, $\mathbf{A} = \mathbf{A}^H$, then the matrix \mathbf{P} can be written as an orthogonal matrix \mathbf{Q} and thus $\mathbf{Q}^{-1} = \mathbf{Q}^H$.⁴
- Using diagonalization one can then show $\mathbf{A}^k = \mathbf{P}\mathbf{D}^k\mathbf{P}^{-1}$ where the elements of \mathbf{D}^k are given by d_{ii}^k . This can be used to efficiently raise a matrix to an unreasonably large power.⁵

Lecture Goals

- Understand when and how the eigenvalue/eigenvector data can be used to construct a diagonal decomposition of a square matrix.
- Notice how self-adjointness of a matrix can be used to make the diagonalization process more efficient.

Lecture Objectives

- State the hypotheses of diagonalization theorem and apply its results to previous eigen-problems for the purposes of defining similarity transformations.
- Using previous problems illustrate how certain matrices can be orthogonally diagonalized.
- Apply the diagonalization procedure to raise matrices to large powers.

⁴If we assume that the algebraic multiplicity of each eigenvalue is 1 then to find \mathbf{Q} we need only normalize the eigenvectors of \mathbf{A} . If this is not the case then orthonormality can be forced by a process known as Gram-Schmidt, which takes linearly independent vectors and creates orthogonal vectors spanning the same space. We will not discuss this here.

⁵At this point one can even consider $\lim_{k \rightarrow \infty} \mathbf{A}^k$.