Homework 2 solutions

1.) $\vec{\nabla}\cdot\dfrac{\hat{\mathscr{r}}}{\mathscr{r}^2}$ $\quad \vec{\mathscr{r}} = \vec{r} - \vec{r}\,' = (x-x')\,\hat{x} + (y-y')\,\hat{y} + (z-z')\,\hat{z}$

$$|\vec{\mathscr{r}}| = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$$

$$\hat{\mathscr{r}} = \frac{\vec{\mathscr{r}}}{|\mathscr{r}|} \qquad \vec{\nabla}\cdot\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = \vec{\nabla}\cdot\frac{\vec{\mathscr{r}}}{|\mathscr{r}|^3} = \underbrace{\frac{\partial}{\partial x}\frac{(x-x')}{|\vec{\mathscr{r}}|^3}}_{\text{term (1)}} + \frac{\partial}{\partial y}\frac{(y-y')}{|\vec{\mathscr{r}}|^3} + \cdots$$

consider term (1)

$$= \frac{1}{|\vec{\mathscr{r}}|^3} + (x-x')\frac{\partial}{\partial x}\frac{1}{[\underbrace{(x-x')^2+(y-y')^2+(z-z')^2}_{|\vec{\mathscr{r}}|^3}]^{3/2}}$$

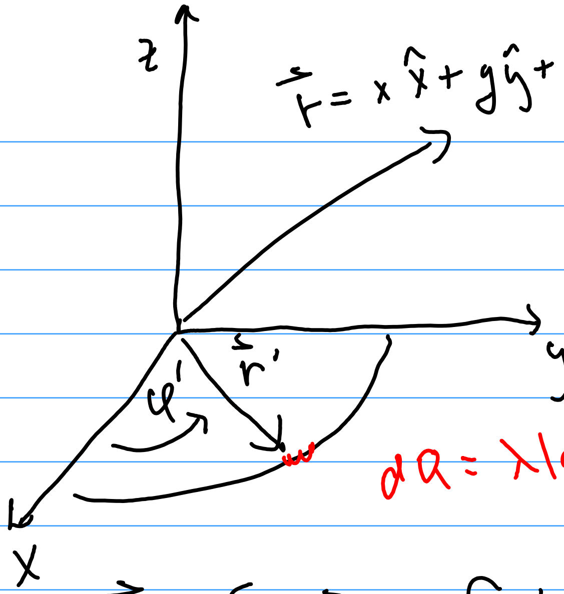
$$= \frac{1}{|\vec{\mathscr{r}}|^3} + \frac{(x-x')(-3/2)\,2(x-x')}{[\quad]^{5/2}} = \frac{1}{[\quad]^{3/2}} - \frac{3(x-x')^2}{[\quad]^{5/2}}$$

term (2) yields $= \dfrac{1}{[\quad]^{3/2}} - \dfrac{3(y-y')^2}{[\quad]^{5/2}}$

term (3) yields $= \underbrace{\dfrac{1}{[\quad]^{3/2}}}_{\text{these add to } \frac{3}{[\quad]^{3/2}}} - \underbrace{\dfrac{3(z-z')^2}{[\quad]^{5/2}}}_{\text{these add to } -\frac{3}{[\quad]^{3/2}}}$

$$\therefore\ \vec{\nabla}\cdot\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = 0$$

2.)



$$\vec r = x\hat x + y\hat y + z\hat z$$

$$\vec r' = R\cos\varphi'\,\hat x + R\sin\varphi'\,\hat y$$

$$\vec{\mathscr{r}} = \vec r - \vec r' \qquad \hat{\mathscr{r}} = \frac{\vec{\mathscr{r}}}{|\vec{\mathscr{r}}|}$$

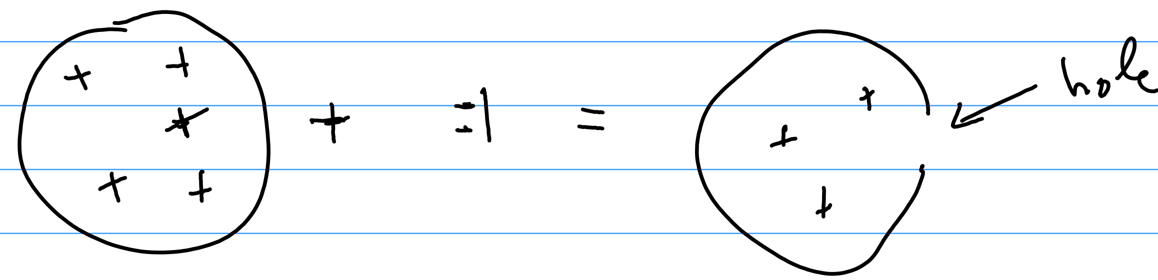
$$dQ = \lambda|d\vec r'| = \lambda R\,d\varphi'$$

$$\vec E = \int d\vec E = \int \frac{k\,dQ}{\mathscr{r}^2}\hat{\mathscr{r}} = \int_0^{\pi} \frac{k\lambda R\,d\varphi'}{\mathscr{r}^2}\hat{\mathscr{r}}$$

3.)

superpose these charge distributions to get the distribution with a hole in it

Q:



this leads to superposing the electric fields to get the electric field with a hole in the sphere

$$\vec E: \quad \frac{\sigma}{\epsilon_0}\hat r + -\frac{\sigma}{2\epsilon_0}\hat r = \frac{\sigma}{2\epsilon_0}\hat r$$

4.) $$\vec\nabla\cdot\vec F = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} = \underbrace{2xz + 1 - 4xz}_{1-2xz}$$

$$x = R\cos\varphi$$

$$\int \vec\nabla\cdot\vec F\,d\tau = \iiint (1 - 2R\cos\varphi\, z)\, r\,d\varphi\,dr\,dz$$

$$d\tau = r\,d\varphi\,dr\,dz$$

$\varphi: 0 \to \pi/2$

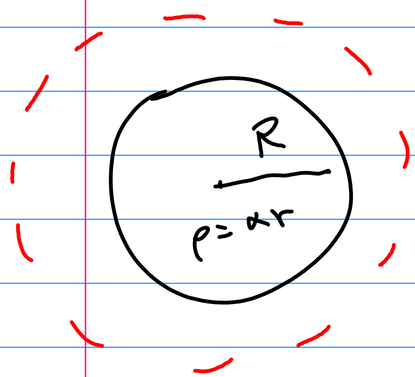
$z: 0 \to 3$

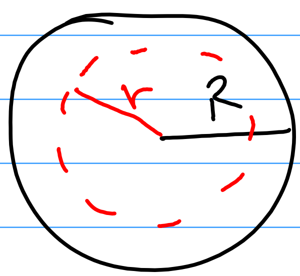
$r: 0 \to 2$

5.)

$$\oint \vec{E}\cdot d\vec{a} = \int \vec{\nabla}\cdot\vec{E}\, d\tau = \underbrace{\int_0^R\int_0^{2\pi}\int_0^{\pi} \alpha r\, r^2 \sin\theta\, d\theta\, d\phi\, dr}_{Q_{enclosed} = \alpha\pi R^4}$$

$$= \rho/\epsilon_0$$





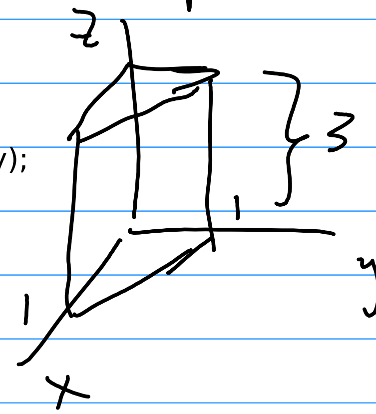
$$\int \rho\, d\tau = \underbrace{\int_0^r\int_0^{2\pi}\int_0^{\pi} \alpha r\, r^2 \sin\theta\, d\theta\, d\phi\, dr} = \alpha\pi r^4 = \oint \vec{E}\cdot d\vec{a}$$

$$\vec{E} = \frac{\alpha\pi r^4}{4\pi r^2 \epsilon_0}\hat{r} = \frac{\alpha r^2}{4\epsilon_0}\hat{r}$$

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{r^2}\frac{\partial}{\partial r} r^2 E_r = \frac{1}{r^2}\frac{\partial}{\partial r}\frac{\alpha r^4}{4\epsilon_0} = \frac{\alpha r}{\epsilon_0} = \frac{\rho}{\epsilon_0}$$

as expected

6.) $\rho = xyz^2 \qquad Q = \int \rho\, dx\, dy\, dz$

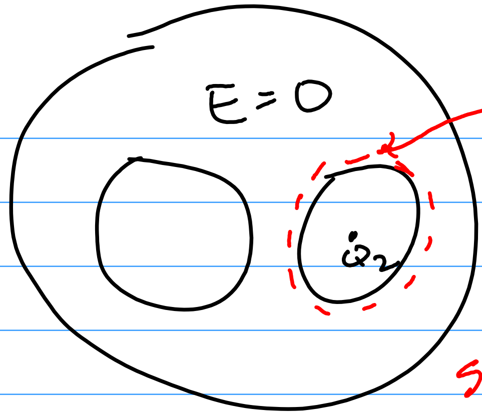


You can do the three integrals in any order. Let's do x first: it runs for 0 to (1-y); then y goes which goes from 0 to 1; and finally z which goes from 0 to 3

$$Q = \int_0^3 z^2 \left\{ \int_0^1 y \left[\int_0^{1-y} x\, dx \right] dy \right\} dz$$

$$= \frac{1}{2}\int_0^3 z^2\, dz \int_0^1 (1-y)^2 y\, dy = 3/8$$

7.)



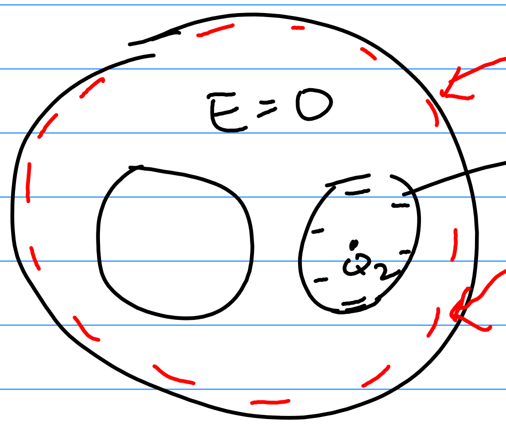
$\oint \vec{E}\cdot d\vec{a} = 0$ just inside the conductor but around the hole since $E = 0$ in conductor

Since Q_2 is enclosed by this surface $-Q_2$ must be on the inner surface of the hole so $Q_{encl} = 0$

This $-Q_2$ must have come from the metal. Since the metal was neutral to start with there must be $+Q_2$ somewhere on the metal.

$+Q_2$ on the metal.

$$\oint \underset{\text{0 in metal}}{\underbrace{\vec{E}}}\cdot d\vec{a} = \frac{Q_{encl}}{\epsilon_0} = 0$$



$-Q_2$

This Gaussian surface shows that $+Q_2$ must reside on the outer surface.

8.) See lecture 7 notes for the answer.