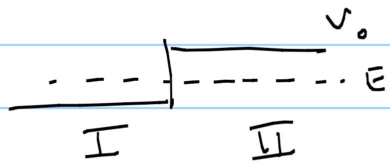
EXAM 2 Solutions



I $\quad -\frac{\hbar^2}{2m}\psi'' = E\psi$

II $\quad -\frac{\hbar^2}{2m}\psi'' + V_0\psi = E\psi$

$\Rightarrow \quad \psi'' + \frac{2mE}{\hbar^2}\psi = 0$

$\Rightarrow \quad \psi'' + 2m(E-V_0)\psi = 0$

since $E < V_0$ rewrite as

$\psi'' - \frac{2m}{\hbar^2}(V_0 - E)\psi = 0$

$\downarrow$

$\psi'' + k^2\psi = 0$

$k^2 = \frac{2mE}{\hbar^2}$

$\psi'' - \kappa^2\psi = 0$

$\kappa^2 = \frac{2m}{\hbar^2}(V_0 - E)$

I $\quad \psi(x) = \underline{Ae^{ikx} + Be^{-ikx}}$

II $\quad \psi(x) = \cancel{Ce^{\kappa x}} + \underline{De^{-\kappa x}}$

blows up

cont. of $\psi(x)$ at $x=0$

$$A + B = D$$

cont of ψ' at $x=0$

$$ik(A-B) = -\kappa D$$

So $\quad A + B = -\frac{ik}{\kappa}(A - B)$

$\Rightarrow \quad A + ik/\kappa = -B + ik/\kappa\, B$

$\Rightarrow \quad \frac{B}{A} = \frac{\kappa + ik}{-\kappa + ik}$

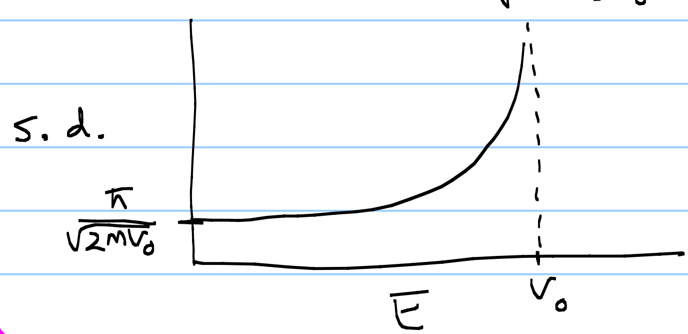
$\boxed{R = 1}$

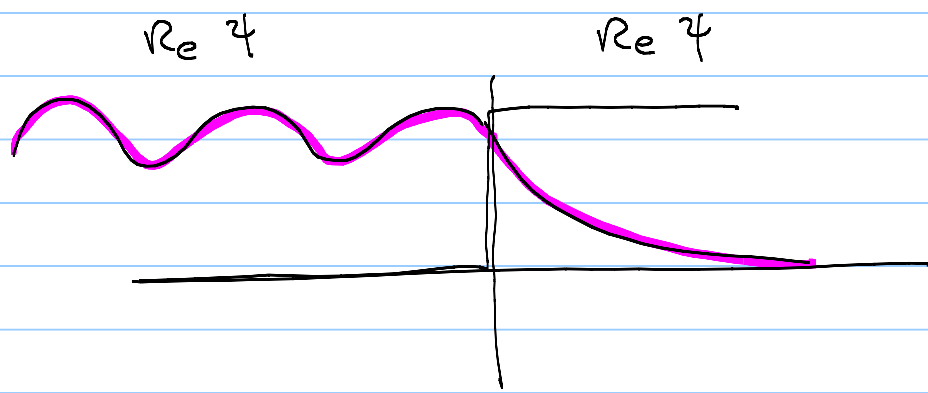
interp.: the probability of a particle being reflected is 1. Any penetration of the wavefcn into the classically forbidden region is exponentially decaying.

in Region

II $\qquad \psi(x) = D e^{-Kx} \qquad K = \dfrac{\sqrt{2m(V_0-E)}}{\hbar}$

$$\boxed{\text{skin depth} = \frac{\hbar}{\sqrt{2m(V_0-E)}}}$$





similar for $|\psi|^2$. It's oscillatory for $x < 0$ and expon. decaying for $x > 0$. Decays faster than Re(ψ)

Be sure you understand why $|\psi|^2$ is oscillatory for $x<0$

$\boxed{2}$ $$\frac{d}{dt}\langle Q\rangle = \frac{i}{\hbar}\langle [H,Q]\rangle + \left\langle \frac{\partial Q}{\partial t}\right\rangle$$

for $Q = x$ $\quad \frac{\partial x}{\partial t} = 0$ so we need

only $[H, x]$.

$$[H,x]\,\varphi(x) = \left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V\right) x\,\varphi(x)$$
$$- x\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V\right)\varphi(x)$$

V commutes with x so

$$-\frac{\hbar^2}{2m}\Big[\underbrace{\frac{d^2}{dx^2}(x\varphi) - x\frac{d^2}{dx^2}\varphi}_{2\frac{d\varphi}{dx}}\Big] = -\frac{\hbar^2}{m}\frac{d}{dx}\varphi$$

So $[H, x]\varphi = -\frac{\hbar^2}{m}\frac{d}{dx}\varphi$ $\qquad p = -i\hbar\frac{d}{dx}$

$$= -\frac{i\hbar}{m}\, p\,\varphi$$

according to the theorem

$$\frac{d}{dt}\langle x\rangle = \frac{i}{\hbar}\langle [H, x]\rangle$$
$$= \frac{i}{\hbar}\left(-\frac{i\hbar}{m}\right)\langle p\rangle$$

$$\boxed{\frac{d}{dt}\langle x\rangle = \langle p\rangle/m}$$

1.33 in Book

3

$$R = \frac{|B|^2}{|A|^2} = \frac{\beta^2}{1+\beta^2}$$

$$T = \frac{1}{1+\beta^2}$$

$$\beta = \frac{m\alpha}{\hbar^2 k} \qquad k = \frac{\sqrt{2mE}}{\hbar}$$

$$\Rightarrow \beta = \frac{m\alpha}{\hbar\sqrt{2mE}} \Rightarrow \beta^2 = \frac{m\alpha^2}{2\hbar^2 E}$$

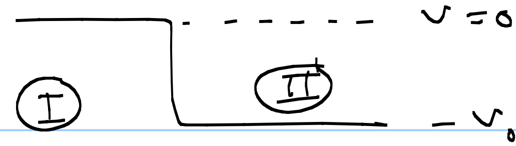
$$T = \frac{1}{1 + m\alpha^2/2\hbar^2 E} = \boxed{\frac{E}{\frac{m\alpha^2}{2\hbar^2} + E}}$$

$$R = 1 - T = \boxed{\frac{m\alpha^2/2\hbar^2}{m\alpha^2/2\hbar^2 + E}}$$

lim $E \to \infty$ $\quad R \to 0 \quad T \to 1$

if $V = \alpha\delta$ R and T do not chance since they depend on α^2.

$\boxed{4}$



I $\quad \psi = A e^{ik_1 x} + B e^{-ik_1 x} \qquad k_1 = \sqrt{2mE}/\hbar$

II $\quad \psi = C e^{ik_2 x} \qquad k_2 = \sqrt{2m(E+V_0)}$

$$A + B = C$$
$$ik_1 (A - B) = ik_2 C$$

!

$$\Rightarrow \quad A + B = \frac{k_1}{k_2}(A - B)$$

$$\Rightarrow \quad A(k_2 - k_1) = -(k_1 + k_2) B$$

$$\Rightarrow \quad \frac{|B|^2}{|A|^2} = \frac{(k_2 - k_1)^2}{(k_1 + k_2)^2} = \frac{(\sqrt{E} - \sqrt{E+V_0})^2}{(\sqrt{E} + \sqrt{E+V_0})^2}$$

$$k_1 = \frac{\sqrt{2mE}}{\hbar} \qquad = \frac{(1 - \sqrt{1 + V_0/E})^2}{(1 + \sqrt{1+V_0/E})^2}$$

$$k_2 = \frac{\sqrt{2m(E+V_0)}}{\hbar} \qquad V_0/E = 3$$

$$\Rightarrow \quad R = \left(\frac{1 - \sqrt{1+3}}{1 + \sqrt{1+3}}\right)^2 = \left(\frac{-1}{3}\right)^2$$

$$\boxed{= \frac{1}{9}}$$

(5)

$$[AB, C] = ABC - CBA$$

$$A[B, C] = ABC - ACB$$

$$[A, C]B = ACB - CAB$$

and these

$$ABC - CAB$$

QED

$$[f(x), p]\,\varphi(x) = f\left(-i\hbar\frac{d}{dx}\right)\varphi - \left(-i\hbar\frac{d}{dx}\right)f\varphi$$

$$= -i\hbar\left(f\varphi' - f'\varphi - f\varphi'\right)$$

$$= +i\hbar f'\phi$$ so

$$[f, p]\,\varphi = i\hbar f'\phi$$

$$\Rightarrow [f, p] = i\hbar f'$$