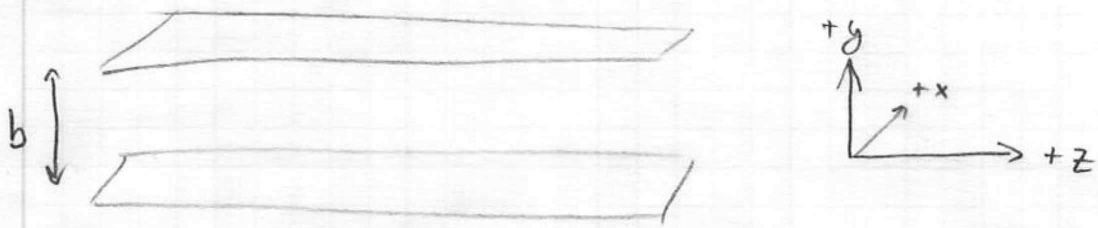
Recitation: TM modes, parallel plate waveguide



(a)

We'll start with a trial B. We'll need something with some spatial variation to get something richer than a TEM solution, and we want something that has a transverse B. The problem is x-invariant, so maybe our B should be x independent also. Something like:

$$\vec{B} = B(y)\, e^{i(kz-\omega t)}\, \hat{\imath}$$

could work. This satisfies $\nabla\cdot\vec{B} = 0$ immediately

Let's check $\nabla\times\vec{B} = \mu_0\varepsilon_0\, \partial\vec{E}/\partial t = \frac{1}{c^2}\,\partial\vec{E}/\partial t$

We need to take the curl of $\vec{B}$. The nonzero terms are:

$$\nabla\times\vec{B} = \frac{\partial B_x}{\partial z}\hat{\jmath} - \frac{\partial B_x}{\partial y}\hat{k} = ik\,B(y)\,e^{i(kz-\omega t)}\,\hat{\jmath} - B'(y)\,e^{i(kz-\omega t)}\,\hat{k}$$

$$\Rightarrow\quad c^2\,\nabla\times\vec{B} = ic^2k\,B(y)\,e^{i(kz-\omega t)}\,\hat{\jmath} - c^2B'(y)\,e^{i(kz-\omega t)}\,\hat{k} = \partial\vec{E}/\partial t$$

So we get E by taking an antiderivative, which is basically division by $-i\omega$:

$$\vec{E} = \frac{ic^2k}{-i\omega}B(y)\,e^{i(kz-\omega t)}\,\hat{\jmath} + \frac{c^2}{i\omega}B'(y)\,e^{i(kz-\omega t)}\,\hat{k} \qquad \text{I}$$

If $\vec{E}$ has this form, $\nabla\times\vec{B} = \frac{1}{c^2}\,\partial\vec{E}/\partial t$ is satisfied.

Let's check $\nabla\times\vec{E} = -\partial\vec{B}/\partial t$. The nonzero $\nabla\times\vec{E}$ terms are:

$$\nabla\times\vec{E} = \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right)\hat{\imath} = \frac{c^2}{i\omega}B''(y)\,e^{i(kz-\omega t)}\,\hat{\imath} - -\frac{c^2k}{\omega}\cdot ik\,B(y)\,e^{i(kz-\omega t)}\,\hat{\imath}$$

$$= -\frac{\partial\vec{B}}{\partial t} = +i\omega\,B(y)\,e^{i(kz-\omega t)}\,\hat{\imath}$$

Grouping up terms, for $\nabla \times \bar{E} = -\partial \bar{B}/\partial t$ to be true, we need:

$$\frac{c^2}{i\omega} B''(y) \cancel{e^{i(kz-\omega t)}} \cancel{\hat{\imath}} + \frac{ic^2k^2}{\omega} B(y) \cancel{e^{i(kz-\omega t)}} \cancel{\hat{\imath}} = +i\omega B(y) e^{i(kz-\omega t)} \hat{\imath}$$

$$\Rightarrow \quad \frac{c^2}{i\omega} B''(y) + \frac{ic^2k^2}{\omega} B(y) = +i\omega B(y)$$

$$\Rightarrow \quad B''(y) + \left(\frac{i\omega}{\cancel{c^2}}\right)\left(\frac{i\cancel{c^2}k^2}{\cancel{\omega}}\right) B(y) = (+i\omega)\cdot\left(\frac{i\omega}{c^2}\right) B(y)$$

$$\Rightarrow \quad B''(y) = \left(-\frac{\omega^2}{c^2} + k^2\right) B(y)$$

$$B''(y) = -\left(\frac{\omega^2}{c^2} - k^2\right) B(y)$$

This is a differential equation for B that has solution

$$B(y) = C_1 \sin(\gamma y) + C_2 \cos(\gamma y) \quad \text{with} \quad \gamma = \sqrt{\omega^2/c^2 - k^2}$$

which gives us a step towards a dispersion relation.

Before we move on, let's check $\nabla \cdot \bar{E} = 0$ using eqn I on the first page. We get:

$$\nabla \cdot \bar{E} = \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = -\frac{c^2 k}{\omega} B'(y) e^{i(kz-\omega t)} + \frac{ikc^2}{i\omega} B'(y) e^{i(kz-\omega t)} = 0$$

So all the Maxwell equations are satisfied subject to the identified constraints.