MACS 332A - Fall 2008

NAME:___________________________________ **Exam I**

In order to receive full credit, SHOW ALL YOUR WORK. Full credit will be given only if all reasoning and work is provided. Please enclose your final answers in boxes.

1. (12 points) Let $A = \begin{bmatrix} 1 & 3 & -2 \\ 2 & 0 & 0 \\ 2 & -1 & 1 \end{bmatrix}$. A has distinct integer eigenvalues. Find the eigenvalues of A.

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 3 & 2 \\ 2 & -\lambda & 0 \\ 2 & -1 & 1-\lambda \end{vmatrix} = -2 \begin{vmatrix} 2 & 2 \\ -\lambda & -1 \end{vmatrix} + (1-\lambda) \begin{vmatrix} 1-\lambda & 2 \\ 3 & -\lambda \end{vmatrix}$$

$$= -2(-2+2\lambda) + (1-\lambda)(\lambda^2 - \lambda - 6) = (1-\lambda)(\lambda - 2)(\lambda + 1)$$

Thus the eigenvalues of A are $\lambda = -1, 1, 2$.

2. Let $A = \begin{bmatrix} -3 & -7 & -5 \\ 2 & 4 & 3 \\ 1 & 2 & 2 \end{bmatrix}$. $\lambda = 1$ is an eigenvalue for A and has a multiplicity of 3.

(a) (10 points) Find a basis for the eigenspace corresponding to $\lambda = 1$.

$$A - \lambda I = \begin{bmatrix} -4 & -7 & -5 \\ 2 & 3 & 3 \\ 1 & 2 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 3 \\ -4 & -7 & -5 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, for the Null space we have $\mathbf{x} = x_3 \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$, giving a basis of $\left\{ \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} \right\}$.

(b) (6 points) If A is diagonalizable, find P and D that diagonalizes A. If A is not diagonalizable, explain why.
A is not diagonalizable since the dimension of the eigenspace corresponding to $\lambda = 1$ is less than the multiplicity of the $\lambda = 1$.

3. (10 points) Use coordinate vectors to determine whether $H = \{2x, x^3 - 3, x - 4x^3, x^3 + 18x - 9\}$ forms a basis for $\mathbb{P}_3$. Justify your answer.
The corresponding coordinate vectors are

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -3 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -4 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 1 \\ 0 \\ 18 \\ -9 \end{bmatrix}$$

If we view these as the columns of a matrix, we see that, due to the row of 0's, we will not have a pivot in every column. Therefore, the columns are linearly dependent and thus, cannot form a basis for $\mathbb{P}_3$.

4. (12 points) The set of all 4×4matrics, $M_{4\times4}$, is a vector space. Prove whether or not that $H = \{D \in M_{4\times4} \ : \ D \text{ is diagonal } \}$ is a subspace of $M_{4\times4}$.

   (a) $H \subseteq M_{4\times4}$: By definition above, we see that H is a subset of $M_{4\times4}$.

   (b) $\mathbf{0} \in H$. In this case $\mathbf{0} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ which is trivially a diagonal matrix and therefore in H.

   (c) H is closed under matrix addition: Let A,$B \in H$

$$\Rightarrow A = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{bmatrix}, B = \begin{bmatrix} w & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & y & 0 \\ 0 & 0 & 0 & x \end{bmatrix} \Rightarrow A{+}B = \begin{bmatrix} a+w & 0 & 0 & 0 \\ 0 & b+x & 0 & 0 \\ 0 & 0 & c+y & 0 \\ 0 & 0 & 0 & d+z \end{bmatrix}$$

   Thus $A + B \in H$

   (d) H is closed under scalar multiplication: Similarly, for $\alpha \in \mathbb{R}$ and $A \in H$, we see that $\alpha A \in H$, since scalar multiplication distributes over the diagonal entries of A and results in a diagonal matrix.

   Thus, H is a subspace of $M_{4\times4}$.

5. Let $A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 2 & 4 & 6 \\ 3 & 3 & 6 \end{bmatrix}$. Find

   (a) (10 points) Nul$A$
   Row reducing A gives

$$A \rightsquigarrow \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 2 & 4 & 6 \\ 3 & 3 & 6 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 4 & 4 \\ 0 & 3 & 3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

   Giving the solution in parametric form,

$$\mathbf{x} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}. \text{ Thus Nul}A = \text{Span}\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

   (b) (5 points) Col$A$ From the work above, we see that Col$A = \text{Span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 4 \\ 3 \end{bmatrix} \right\}$

   (c) (3 points) dim Row$A$ = dim Col$A = 2$ (based on the information found in (b))

   (d) (3 points) rank$A$ = dim Col$A = 2$.

6. Find a basis in $\mathbb{R}^3$ for the set of vectors in the plane $2x_1 - 3x_2 + 6x_3 = 0$.
In matrix form, the system representing the plane can be written as

$$A\mathbf{x} = \mathbf{0} \text{ where } A = \begin{bmatrix} 2 & -3 & 6 \end{bmatrix}$$

In parametric form, we then have $\mathbf{x} = x_2 \begin{bmatrix} 3/2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$

Thus a basis for this plane is $\left\{ \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$

7. (12 points) Let $A = \begin{bmatrix} 5 & -1 \\ 5 & 1 \end{bmatrix}$. Find P and C where C has the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ and such that $A = PCP^{-1}$.
$\det(A - \lambda I) = \lambda^2 - 6\lambda + 10$ giving eigenvalues $\lambda = 3 \pm i$
For $\lambda = 3 - i$ we have the corresponding eigenvector $\mathbf{x} = \begin{bmatrix} 2 - i \\ 5 \end{bmatrix}$. Therefore, using the real and imaginary portions of our eigenvalue to form C and the corresponding real and imaginary portions of our eigenvector to form the columns of P, we get

$$C = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}, P = \begin{bmatrix} 2 & -1 \\ 5 & 0 \end{bmatrix}$$

8. (12 points) The President of the United States tells person A his intention to run or not to run in the next election. Then A relays the news to B, who in turn relays the message to C, and so forth, always to some new person. We assume that there is a probability of 0.4 that a person will change the answer from yes to no when transmitting it to the next person and a probability of 0.3 that he or she will change it from no to yes.

   (a) Set up a stochastic matrix for this problem.

   Based on the percentages given, the corresponding stochastic matrix is $S = \begin{bmatrix} .6 & .3 \\ .4 & .7 \end{bmatrix}$

   (b) If the president initially says yes he will run, what is the probability that C will hear yes from B?

   Since the president initially says he will run, our initial probability vector is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.
   Applying our stochastic matrix from above first to simulate A to B and then again to simulate B to C, we have

$$\begin{bmatrix} .6 & .3 \\ .4 & .7 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} .6 \\ .4 \end{bmatrix} \rightsquigarrow \begin{bmatrix} .6 & .3 \\ .4 & .7 \end{bmatrix} \begin{bmatrix} .6 \\ .4 \end{bmatrix} = \begin{bmatrix} .48 \\ .52 \end{bmatrix}$$