MATH 332 - Linear Algebra
Homework 4, Summer 2009

June 29, 2009
Due: July 2, 2009

Matrix Equations - Matrix Inversion - Invertible Matrix Theorem - Matrix Partitioning - Matrix Factorization

1. Given,

$$\mathbf{A}(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}, \quad \theta \in (0, 2\pi]. \tag{1}$$

We now consider the *action* of $\mathbf{A}$ on vectors from $\mathbb{R}^2$. That is, we wish to study the effect of the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ represented by the matrix $\mathbf{A}(\theta)$ where $\theta \in (0, 2\pi]$.

(a) First show that the transformation is one-to-one. [1]

(b) Given this matrix representation of T find the matrix representation of the inverse transformation. That is find $\mathbf{A}^{-1}$.

(c) Let $\mathbf{x} = [1\ 0]^{\mathrm{T}}$. Describe or draw the action of the linear transformation $\mathbf{Ax}$ for $\theta \in S$ where $S = \left\{0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}\right\}$. What would the action of $\mathbf{A}^{-1}$ be?

(d) Let $\mathbf{A}(\theta)\mathbf{x} = \mathbf{b}$ for each $\theta \in S$. Calculate, $\frac{\mathbf{x} \cdot \mathbf{b}}{|\mathbf{x}||\mathbf{b}|}$.[2] How is this related to θ? [3]

(e) If we define the derivative of a matrix function as a matrix of derivatives then a typical product rule results. That is, if $\mathbf{A}, \mathbf{B}$ have elements, which are functions of θ then $\frac{d\,[\mathbf{AB}]}{d\theta} = \mathbf{A}\frac{d\mathbf{B}}{d\theta} + \frac{d\mathbf{A}}{d\theta}\mathbf{B}$. [4] Using this and the identity $\mathbf{AA}^{-1} = \mathbf{I}$ to prove that $\frac{d\left[\mathbf{A}^{-1}\right]}{d\theta} = -\mathbf{A}^{-1}\frac{d\,[\mathbf{A}]}{d\theta}\mathbf{A}^{-1}$. Verify this formula using the matrix given above.

2. Given,

$$\mathbf{A} = \begin{bmatrix} 3 & 6 & 7 \\ 0 & 2 & 1 \\ 2 & 3 & 4 \end{bmatrix}.$$

(a) Calculate $\mathbf{A}^{-1}$ and check your result with the appropriate matrix multiplication.

(b) Let $\mathbf{A}^{-1}_{\text{left}} = (\mathbf{A}^{\mathrm{T}}\mathbf{A})^{-1}\mathbf{A}^{\mathrm{T}}$. Prove that $\mathbf{A}^{-1}_{\text{left}}$ exists and show that $\mathbf{A}^{-1}_{\text{left}}\mathbf{A} = \mathbf{I}$. [5]

(c) Let $\mathbf{A}^{-1}_{\text{right}} = \mathbf{A}^{\mathrm{T}}(\mathbf{AA}^{\mathrm{T}})^{-1}$. Prove that $\mathbf{A}^{-1}_{\text{right}}$ exists and show that $\mathbf{AA}^{-1}_{\text{right}} = \mathbf{I}$.[6]

(d) Let $\mathbf{A}_1 = [2\ 2]^{\mathrm{T}}$ and $\mathbf{A}_2 = [2\ 2]$. Using the previous formula find the left-inverse of $\mathbf{A}_1$ and the right-inverse of $\mathbf{A}_2$. Check your results with the appropriate multiplication.

[1] Recall that a transformation is one-to-one if and only if $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution.

[2] Recall that $\mathbf{x} \cdot \mathbf{y}$ and $|\mathbf{x}|$ are the standard dot-product and magnitude, respectively, from vector-calculus. These operations hold for vectors in $\mathbb{R}^n$ ıt now have the following definitions, $\mathbf{x} \cdot \mathbf{y} = \mathbf{x}^{\mathrm{T}}\mathbf{y}$ and $|\mathbf{x} = \sqrt{\mathbf{x}^{\mathrm{T}}\mathbf{x}}$.

[3] What we are trying to extract here is the standard result from calculus, which relates the dot-product or inner-product on vectors to the angle ətween them. This is clear when we have vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$ since we have tools from trigonometry and geometry but when treating vectors in $^n, n \geq 4$ these tools are no longer available. However, we would still like to have similar results to those of $\mathbb{R}^n, n = 2, 3$. To make a long story short, e will have these results for arbitrary vectors in $\mathbb{R}^n$ but not immediately. The first thing we must do is show that $|\mathbf{x} \cdot \mathbf{y}| \leq |\mathbf{x}||\mathbf{y}|$, which is known as :hwarz's inequality. Without this we cannot be permitted to always relate $\frac{\mathbf{x} \cdot \mathbf{b}}{|\mathbf{x}||\mathbf{b}|}$ to θ via inverse trigonometric functions. These details will occur in ıapter 6 where we find that by using the inner-product on vectors from $\mathbb{R}^n$ we will define the notion of angle and from that distance. Using these efinitions and Schwarz's inequality will then give us a triangle-inequality for arbitrary finite-dimensional vectors. This is to say that the algebra of ctors in $\mathbb{R}^n$ carries its own definition of angle and length - very nice of it don't you think? Also, it should be noted that these results exist for certain -called infinite-dimensional spaces, but are harder to prove - of course, and that the study of linear transformations of such spaces is the general tting for quantum mechanics - see MATH503:Functional Analysis for more details.

[4] To see why this is true differentiate an arbitrary element of $\mathbf{AB}$ to find $\frac{d}{d\theta}[\mathbf{AB}]_{ij} = \frac{d}{d\theta}\sum_{k=1}^{n} a_{ik}b_{kj} = \sum_{k=1}^{n} \frac{da_{ik}}{d\theta}b_{kj} + a_{ik}\frac{db_{kj}}{d\theta}$.

[5] This matrix is called the left-inverse of $\mathbf{A}$ and it can be shown that if $\mathbf{A} \in \mathbb{R}^{m \times n}$ such that $\mathbf{A}$ has a pivot in every column then the left inverse :ists.

[6] This matrix is called the right-inverse of $\mathbf{A}$ and it can be shown that if $\mathbf{A} \in \mathbb{R}^{m \times n}$ such that $\mathbf{A}$ has a pivot in every row then the right inverse :ists.

3. Noting any theorems used from class or the text, prove the following statements:

   (a) If $\mathbf{A}$ is an $n \times n$ matrix and $\mathbf{A}^{-1}$ exists, then the columns of $\mathbf{A}$ span $\mathbb{R}^n$.

   (b) If $\mathbf{A}$ is an $n \times n$ matrix and $\mathbf{Ax} = \mathbf{b}$ has a solution for each $\mathbf{b} \in \mathbb{R}^n$, then $\mathbf{A}$ is invertible.

   (c) If the matrix $\mathbf{A}$ is invertible, then the columns of $\mathbf{A}^{-1}$ are linearly independent.

   (d) If the equation $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A} \in \mathbb{R}^{n\times n}$, has more than one solution for some $\mathbf{b} \in \mathbb{R}^n$, then the columns of $\mathbf{A}$ do not span $\mathbb{R}^n$.

   (e) If the equation $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A} \in \mathbb{R}^{n\times n}$, is inconsistent for some $\mathbf{b} \in \mathbb{R}^n$, then the equation $\mathbf{Ax} = \mathbf{0}$ has a non-trivial solution.

   (f) If $\mathbf{A}$ is a square matrix with two identical columns then $\mathbf{A}^{-1}$ does not exist.

4. Suppose that $\mathbf{A} \in \mathbb{R}^{n\times n}$ is written in partitioned form as,

$$\mathbf{A} = \begin{bmatrix} \mathbf{P} & \mathbf{Q} \\ \mathbf{R} & \mathbf{S} \end{bmatrix}. \tag{2}$$

   (a) Suppose that $\mathbf{A}$ and $\mathbf{P}$ are non-singular and prove that,

$$\mathbf{A}^{-1} = \begin{bmatrix} \mathbf{X} & -\mathbf{P}^{-1}\mathbf{QW} \\ -\mathbf{WRP}^{-1} & \mathbf{W} \end{bmatrix}, \tag{3}$$

   where $\mathbf{W} = (\mathbf{S} - \mathbf{RP}^{-1}\mathbf{Q})^{-1}$ and $\mathbf{X} = \mathbf{P}^{-1} + \mathbf{P}^{-1}\mathbf{QWRP}^{-1}$.

   (b) Suppose that $\mathbf{A}$ and $\mathbf{S}$ are non-singular and prove that,

$$\mathbf{A}^{-1} = \begin{bmatrix} \mathbf{X} & -\mathbf{XQS}^{-1} \\ -\mathbf{S}^{-1}\mathbf{RX} & \mathbf{W} \end{bmatrix}, \tag{4}$$

   where $\mathbf{X} = (\mathbf{P} - \mathbf{QS}^{-1}\mathbf{R})^{-1}$ and $\mathbf{W} = \mathbf{S}^{-1} + \mathbf{S}^{-1}\mathbf{RXQS}^{-1}$.

   (c) Show that if $\mathbf{P}, \mathbf{S}, \mathbf{A}$ are all non-singular matrices then the two previous forms are equivalent and that $(\mathbf{S}-\mathbf{RP}^{-1}\mathbf{Q})^{-1} = \mathbf{S}^{-1} + \mathbf{S}^{-1}\mathbf{RXQS}^{-1}$.

   (d) Finally, test these previous formula for $\mathbf{P} = a$, $\mathbf{Q} = b$, $\mathbf{R} = c$, $\mathbf{S} = d$ where $a, b, c, d \in \mathbb{R}$ such that $ad - bc \neq 0$.

5. Determine the LU-Decomposition of the matrix $\mathbf{A}$ **and** check your result for $\mathbf{L}$ by multiplication of three elementary matrices.

$$\mathbf{A} = \begin{bmatrix} 1 & 4 & -1 & 5 \\ 3 & 7 & -2 & 9 \\ -2 & -3 & 1 & -4 \end{bmatrix}$$

   Hint: The matrix $\mathbf{U}$, found by three steps of row reduction on $\mathbf{A}$, will have two pivot columns. These two pivot columns are used to determine the first two columns of $\mathbf{L}_{3\times3}$. The remaining column of $\mathbf{L}$ is equal the last column of $\mathbf{I}_3$.

1) Given,

$$A(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

a) $A(\theta)\vec{x} = \vec{0} \iff \left[\begin{array}{cc|c} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \end{array}\right] \sim$

$$\sim \left[\begin{array}{cc|c} \cos(\theta)\sin(\theta) & -\sin^2(\theta) & 0 \\ \sin(\theta)\cos(\theta) & \cos^2(\theta) & 0 \end{array}\right] \sim$$

$$\sim \left[\begin{array}{cc|c} \cos(\theta)\sin(\theta) & -\sin^2(\theta) & 0 \\ 0 & \underbrace{\cos^2(\theta) + \sin^2(\theta)}_{1} & 0 \end{array}\right]$$

$\Rightarrow x_2 = 0 \Rightarrow x_1 = 0 \Rightarrow \vec{x} = \vec{0}$ is the only Soln

$\Rightarrow$ A is a 1 to 1 linear transformation.

b) By theorem 2.2.4 we have,

$$A^{-1} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \cdot \frac{1}{\cos^2\theta + \sin^2\theta} \overset{\text{note}}{=} A^T$$

c) Case $\theta = 0$.

$$A\vec{x} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\theta = \frac{\pi}{6}$

$$\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix}$$

These transformations have Rotated $\vec{x}$ Counter clockwise along the unit circle by an angle of θ.

$\theta = \frac{\pi}{3}$

$$\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ \sqrt{3}/2 \end{bmatrix}$$

$\theta = \frac{\pi}{2}$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

d) Case $\theta = 0$:

$$\frac{\vec{x}\cdot\vec{b}}{|\vec{x}||\vec{b}|} = \frac{[1 \; 0]\begin{bmatrix} 1 \\ 0 \end{bmatrix}}{1 \cdot 1} = 1 = \cos(\theta)$$

Case $\theta = \frac{\pi}{6}$.

$$\frac{\vec{x}\cdot\vec{b}}{|\vec{x}||\vec{b}|} = \frac{[1\ 0]\begin{bmatrix}\sqrt{3}/2\\ 1/2\end{bmatrix}}{\cancel{|\vec{x}||\vec{b}|}_{\,1}} = \frac{\sqrt{3}}{2} = \cos(\theta)$$

Case $\theta = \frac{\pi}{3}$

$$\frac{\vec{x}\cdot\vec{b}}{|\vec{x}||\vec{b}|} = \frac{1}{2} = \cos(\theta)$$

Case $\theta = \frac{\pi}{2}$

$$\frac{\vec{x}\cdot\vec{b}}{|\vec{x}||\vec{b}|} = 0 = \cos(\theta)$$

From this we notice,

$$\frac{\vec{x}^T\vec{b}}{|\vec{x}||\vec{b}|} = \cos(\theta), \quad \theta \text{ is angle between } \vec{x}, \vec{b}.$$

e) Consider,

$$\frac{d}{d\theta}[AA^{-1}] = A\frac{d[A^{-1}]}{d\theta} + \frac{d[A]}{d\theta}A^{-1} = \frac{d[I]}{d\theta} = 0$$

$$\Rightarrow \quad \frac{d[A^{-1}]}{d\theta} = -A^{-1}\frac{dA}{d\theta}A^{-1}$$

$$\frac{dA^{-1}}{d\theta} = -\begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} -\sin(\theta) & -\cos(\theta) \\ +\cos(\theta) & -\sin(\theta) \end{bmatrix} \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} =$$

$$= -\begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -\sin(\theta) & +\cos(\theta) \\ -\cos(\theta) & \mp\sin(\theta) \end{bmatrix} =$$ [illegible]

b) The Gauss-Jordan Method

$$\left[\begin{array}{ccc|ccc} 3 & 6 & 7 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 2 & 3 & 4 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R1-3R2 \\ \\ 2R1-3R3 \end{array} \left[\begin{array}{ccc|ccc} 3 & 0 & 4 & 1 & -3 & 0 \\ 0 & 2 & 1 & & 0 & 10 \\ 0 & 3 & 2 & 2 & 0 & -3 \end{array}\right] \begin{array}{l} \\ \\ 2R3-3R2 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 3 & 0 & 4 & 1 & -3 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 4 & -3 & -6 \end{array}\right] . \begin{array}{l} R1-4R3 \\ R2-R3 \\ \\ \end{array} \left[\begin{array}{ccc|ccc} 3 & 0 & 0 & -15 & -9 & 24 \\ 0 & 2 & 0 & -4 & 4 & 6 \\ 0 & 0 & 1 & 4 & -3 & -6 \end{array}\right] \begin{array}{l} \div 3 \\ \div 2 \\ . \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -5 & 3 & 8 \\ 0 & 1 & 0 & -2 & 2 & 3 \\ 0 & 0 & 1 & 4 & -3 & -6 \end{array}\right] \qquad A^{-1} = \begin{bmatrix} -5 & 3 & 8 \\ -2 & 2 & 3 \\ 4 & -3 & -6 \end{bmatrix}$$

c) The Cofactor Representation

$$A^{-1} = \frac{1}{det(A)} \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} = \frac{1}{-1} \begin{bmatrix} 5 & -3 & 8 \\ 2 & -2 & -3 \\ -4 & 3 & 6 \end{bmatrix} = \begin{bmatrix} -5 & 3 & 8 \\ -2 & 2 & 3 \\ 4 & -3 & -6 \end{bmatrix}$$

d) Check your result by showing $AA^{-1} = I$

$$AA^{-1} = \begin{bmatrix} 3 & 6 & 7 \\ 0 & 3 & 1 \\ 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} -5 & 3 & 8 \\ -2 & 2 & 3 \\ 4 & -3 & -6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

4.

i. $det(A) = ad - bc$

ii. $det(B) = cb - ad = -(ad - bc) = -det(A)$

iii. $det(D) = d(a + kc) - c(b + kd) = ad + kdc - cb - cdk = ad - bc = det(A)$

iv. $det(C) = adk - kcb = k(ad - bc) = k \cdot det(A)$

$A \sim B$ by a row interchange and ii shows $det(A) = -det(B)$
$A \sim C$ by a row scaling and iv shows $det(A) = k \cdot det(C)$
$A \sim D$ by a row interchange where a multiple of one row is added to another. iii shows that $det(A) = det(D)$

5.

Forward Direction: Assume $A_{3\times x}$ is such that $det(A) = 0$. then the volume of the parallelopiped spaned by $\vec{a}_1, \vec{a}_2, \vec{a}_3$ has zero volume. That is, the parallelopiped does not exist. This implies that all the vectors $\vec{a}_1, \vec{a}_2, \vec{a}_3$ lie in the same plane, and from a linearly dependent set. Thus, by the invertable matrix theorem A^{-1} does not exist.

Backward Direction: Assume A is not invertable. Then the columns of A are linearly dependent and $span\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$ is at most (in terms of dimension) a plane which has zero volume and cannot form a parallelopiped. Thus, $det(A) = 0$.

Define:

b) $A_{left}^{-1} = A_L^{-1} = (A^T A)^{-1} A^T$

if A^{-1} exists then $(A^T)^{-1}$ exists $\Rightarrow (A^T A)^{-1}$ Exists.

and

$$A_L^{-1} A = (A^T A)^{-1} A^T A = A^{-1} (A^T)^{-1} A^T A = A^{-1} I A = I$$

c) $A_R^{-1} = A^T (A A^T)^{-1}$ Exists by similar arguments.

$$A A_R^{-1} = A(A^T (A A^T)^{-1}) = A A^T (A^T)^{-1} A^{-1} = I$$

d) Given $A_1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ then $A_L^{-1} = \left([2\ 2] \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right)^{-1} [2\ 2] = \frac{1}{8}[2\ 2]$

and $A_L^{-1} A_1 = \frac{1}{8}[2\ 2] \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \frac{1}{8} \cdot 8 = 1$

Also, $A_2 = [2\ 2] \Rightarrow A_R^{-1} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \left([2\ 2] \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right)^{-1} = \frac{1}{8} \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

and

$$[2\ 2] \cdot \frac{1}{8} \cdot \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \frac{1}{8} [2\ 2] \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \underline{1}$$

5. a.

Proof

Since A is $n \times n$ and A^{-1} Exist we have that there Exists a solution to

$$A\vec{x} = \vec{b}\ ,\quad \vec{x} \in \mathbb{R}^n$$

for every $\vec{b} \in \mathbb{R}^n$, namely $\vec{x} = A^{-1}\vec{b}$. Thus by theorem 4 we have that the columns of A span $\mathbb{R}^n$. ☒

b.

We have that A is $n \times n$ and

$$A\vec{x} = \vec{b}\ ,\quad x \in \mathbb{R}^n$$

has a solution for every $\vec{b} \in \mathbb{R}^n$. Thus, theorem 4 says that Every Row of A must contain a pivot.

Since A is $n \times n$ Each Column has a pivot. Hence the Reduced Row Echelon form must be a diagonal matrix with 1's along the diagonal. This implies that $A \sim I_n$ and by Theorem 7 A must be invertable. ☒

$7/7$

3.

a. If the matrix A is inventable then A^{-1} is inventable by Theorem 2.8.6. If A^{-1} is inventable then by the inventable matrix theorem it's columns are linearly independent.

b. If $A\vec{x}=\vec{b}$ has more than 1 solution for some $\vec{b}\in\mathbb{R}^n$, then A does not have a pivot for each Row (Presence of a free Variable). Since A does not have a pivot for each row then by the IMT the columns of A do not span $\mathbb{R}^n$.

c. Note - this problem originally contained the typo,

"$A\vec{x}=\vec{b}$ is _consistent for some $\vec{b}\in\mathbb{R}^n$."

It should read inconsistent. We analyze all cases here.

If the problem read "consistent", there would, naturally, be two cases.

Case 1: $A\vec{x}=\vec{b}$ is consistent with <u>nonunique</u> solutions for some $\vec{b}\in\mathbb{R}^n$.

If $A\vec{x}=\vec{b}$ is consistent with nonunique solns for some $\vec{b}\in\mathbb{R}^n$ then there does <u>not</u> Exist a pivot for each Row of A and by the IMT the columns of A do <u>not</u> span $\mathbb{R}^n$.

Case 2:

If $A\vec{x}=\vec{b}$ is consistent with a unique soln for some $\vec{b}\in\mathbb{R}^n$ then A has a pivot for Each Row and by IMT the columns of A do span $\mathbb{R}^n$.

In this case the statement in 3c is false!

· If the problem Read "inconsistent" there is only 1 case whose proof is identical to Case 1 above. It is worth saying that in this case the statement 3c. is always true

d. If A is a square matrix with 2 identical columns then the columns of A are linearly dependent and by IMT A is not invertable. Hence, A^{-1} does not Exist.

4/7

Let $A = \begin{bmatrix} P & Q \\ R & S \end{bmatrix}$

a) Suppose A, P are nonsingular then A^{-1} Exist s.t.

$$A^{-1} = \begin{bmatrix} X & -P^{-1}QW \\ -WRP^{-1} & W \end{bmatrix}$$

where $W = (S - RP^{-1}Q)^{-1}$ and $X = P^{-1} + P^{-1}QWRP^{-1}$.

Proof:

$$[AA^{-1}]_{11} = PX - QWRP^{-1} =$$
$$= P(P^{-1} + P^{-1}QWRP^{-1}) - QWRP^{-1} = I + QWRP^{-1} - QWRP^{-1} = I$$

$$[AA^{-1}]_{12} = -PP^{-1}QW + QW = 0$$

$$[AA^{-1}]_{21} = RX - SWRP^{-1} = R(P^{-1} + P^{-1}QWRP^{-1}) - (W^{-1} + RP^{-1}Q)WRP^{-1} =$$
$$= RP^{-1} + RP^{-1}QWRP^{-1} - \underbrace{W^{-1}W}_{I}RP^{-1} - RP^{-1}QWRP^{-1} = 0$$

$$[AA^{-1}]_{22} = -RP^{-1}QW + SW = -RP^{-1}QW + (W^{-1} + RP^{-1}Q)W =$$
$$= -RP^{-1}QW + I + RP^{-1}QW = I$$

$$\Rightarrow \quad AA^{-1} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$, which is the $n \times n$ Identity matrix.

b) Suppose A, S are nonsingular then A^{-1} exists s.t.

$$A^{-1} = \begin{bmatrix} X & -XQS^{-1} \\ -S^{-1}RX & W \end{bmatrix}$$

where $X = (P - QS^{-1}R)^{-1}$, $W = S^{-1} + S^{-1}RXQS^{-1}$

$$[A^{-1}A]_{11} = XP - XQS^{-1}R =$$

$$= X(X^{-1} + QS^{-1}R) - XQS^{-1}R = I + XQS^{-1}R - XQS^{-1}R = I$$

$$[A^{-1}A]_{12} = XQ - XQ\cancel{S^{-1}S}^{I} = 0$$

$$[A^{-1}A]_{21} = -S^{-1}RXP + WR = -S^{-1}R\cancel{X}(X^{-1} + QS^{-1}R) + (S^{-1} + S^{-1}RXQS^{-1})R =$$

$$= -S^{-1}R + -S^{-1}RQS^{-1}R + S^{-1}R + S^{-1}RXQS^{-1}R = 0$$

$$[A^{-1}A]_{22} = -S^{-1}RXQ + WS = -S^{-1}RXQ + (S^{-1} + S^{-1}RXQS^{-1})S =$$

$$= -S^{-1}RXQ + I + S^{-1}RXQ\cancel{S^{-1}S}^{I} = I$$

$$\Rightarrow [A^{-1}A] = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$ which is the $n \times n$ Id. matrix.

c) If P, S, A are nonsingular then all previous matrices are defined. Since A^{-1} is ~~exists~~ unique the two previous matrices are equal, which implies

$-P^{-1}QW = -XQS^{-1}$ and $-WRP^{-1} = -S^{-1}RX$

$X = P^{-1} + P^{-1}QWRP^{-1} = (P - QS^{-1}R)^{-1}$

$W = (S - RP^{-1}Q)^{-1} = S^{-1} + S^{-1}RXQS^{-1}$

a) To test the formula,

$P = a,\ Q = b,\ R = c,\ S = d$

$\Rightarrow X = P^{-1} + P^{-1}QWRP^{-1} = \frac{1}{a} + \frac{1}{a} \cdot b \cdot W \cdot c \cdot \frac{1}{a}$

where $W = \left(d - \frac{c}{a} \cdot b\right)^{-1} = \frac{a}{ad-bc}$

$\Rightarrow X = \frac{1}{a} + \frac{1}{a} b \cdot \frac{a}{ad-bc} \cdot c \cdot \frac{1}{a} =$

$= \frac{1}{a} + \frac{bc}{a(ad-bc)} = \frac{a(ad-bc)+abc}{a(ad-bc)} = \frac{d}{ad-bc}$

also $= \frac{1}{a}\left(1 + \frac{bc}{ad-bc}\right) = \frac{1}{a}\left(\frac{ad-bc+bc}{ad-bc}\right)$

$-P^{-1}QW = -\frac{1}{a} \cdot b \cdot \frac{a}{ad-bc} = \frac{-b}{ad-bc}$

and

$-WRP^{-1} = \frac{-a}{ad-bc} \cdot c \cdot \frac{1}{a} = \frac{-c}{ad-bc}$

$\Rightarrow A^{-1} = \begin{bmatrix} X & -P^{-1}QW \\ -WRP^{-1} & W \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

which is the inverse of

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

2. $A=\begin{bmatrix}1 & 4 & -1 & 5\\ 3 & 7 & -2 & 9\\ -2 & -3 & 1 & -4\end{bmatrix} \underset{R3=R3+2R1}{\overset{R2=R2-3R1}{\sim}} \begin{bmatrix}1 & 4 & -1 & 5\\ 0 & -5 & 1 & -6\\ 0 & 5 & -1 & 6\end{bmatrix} \overset{R3=R3+R2}{\sim}$

$\sim \begin{bmatrix}1 & 4 & -1 & 5\\ 0 & -5 & 1 & -6\\ 0 & 0 & 0 & 0\end{bmatrix} = U, \quad L=\begin{bmatrix}1 & 0 & 0\\ 3 & 1 & 0\\ -2 & 3 & 1\end{bmatrix}$ (By Quick Method)

$E_1^{-1}=\begin{bmatrix}1 & 0 & 0\\ +3 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}, \quad E_2^{-1}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ -2 & 0 & 1\end{bmatrix}, \quad E_3^{-1}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & -1 & 1\end{bmatrix}$

$E_1^{-1}E_2^{-1}E_3^{-1} \quad \begin{bmatrix}1 & 0 & 0\\ 3 & 1 & 0\\ -2 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & -1 & 1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 3 & 1 & 0\\ -2 & -1 & 1\end{bmatrix} = L$

$LU=\begin{bmatrix}1 & 0 & 0\\ 3 & 1 & 0\\ -2 & -1 & 1\end{bmatrix}\begin{bmatrix}1 & 4 & -1 & 5\\ 0 & -5 & 1 & -6\\ 0 & 0 & 0 & 0\end{bmatrix}=\begin{bmatrix}1 & 4 & -1 & 5\\ 3 & 7 & -2 & 9\\ -2 & -3 & 1 & -4\end{bmatrix} = A.$

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