

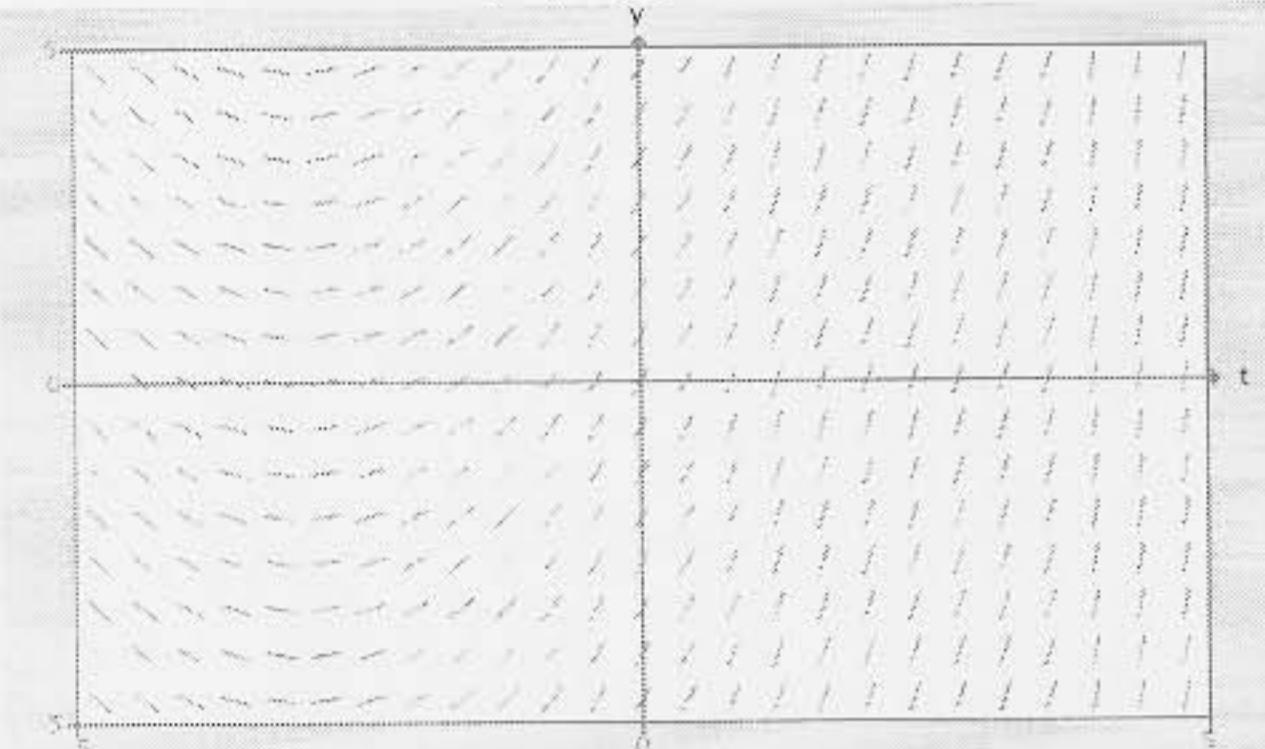
In order to receive full credit, SHOW ALL YOUR WORK. Full credit will be given only if all reasoning and work is provided. Where appropriate, please enclose your final answers in boxes.

1. (8 points) Match the following slope fields to their differential equations.

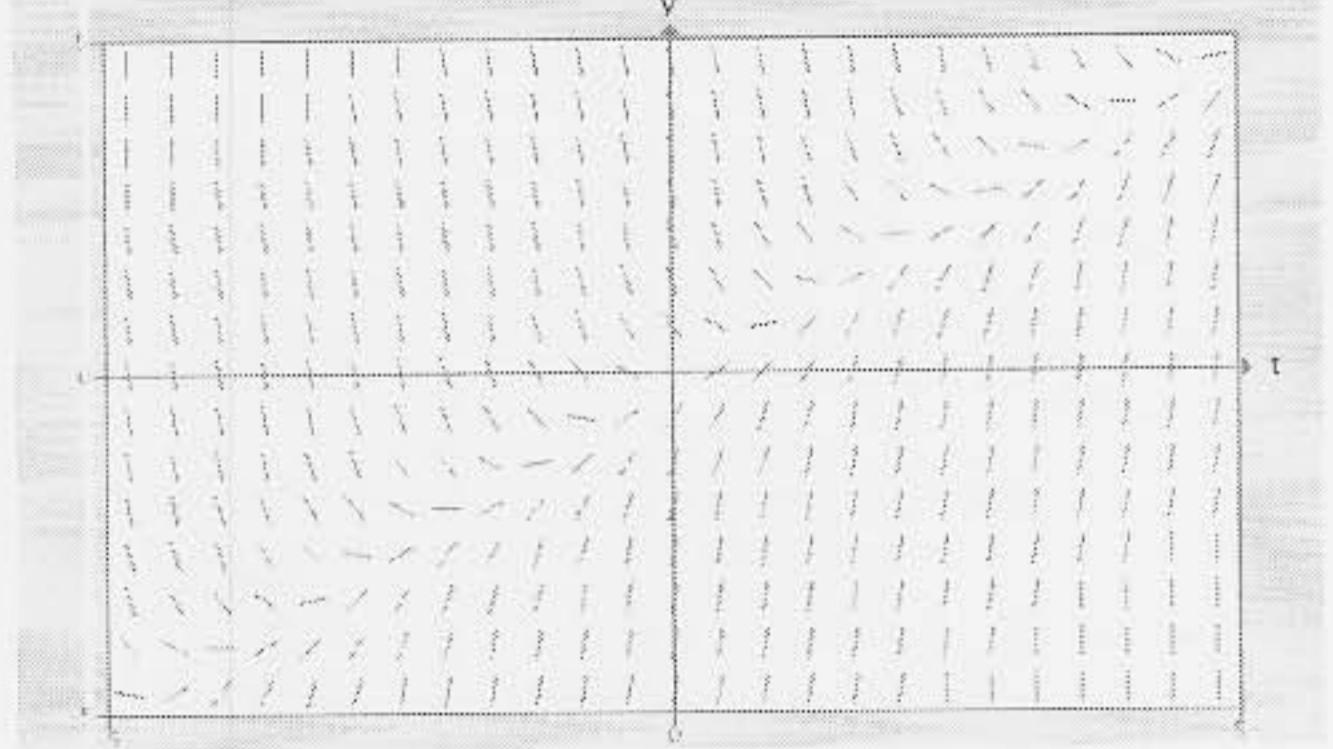
(i)  $\frac{dy}{dt} = 3 - y$     (ii)  $\frac{dy}{dt} = 3 + y$     (iii)  $\frac{dy}{dt} = 3 - t$

(iv)  $\frac{dy}{dt} = 3 + t$     (v)  $\frac{dy}{dt} = 3ty$     (vi)  $\frac{dy}{dt} = 3(t - y)$

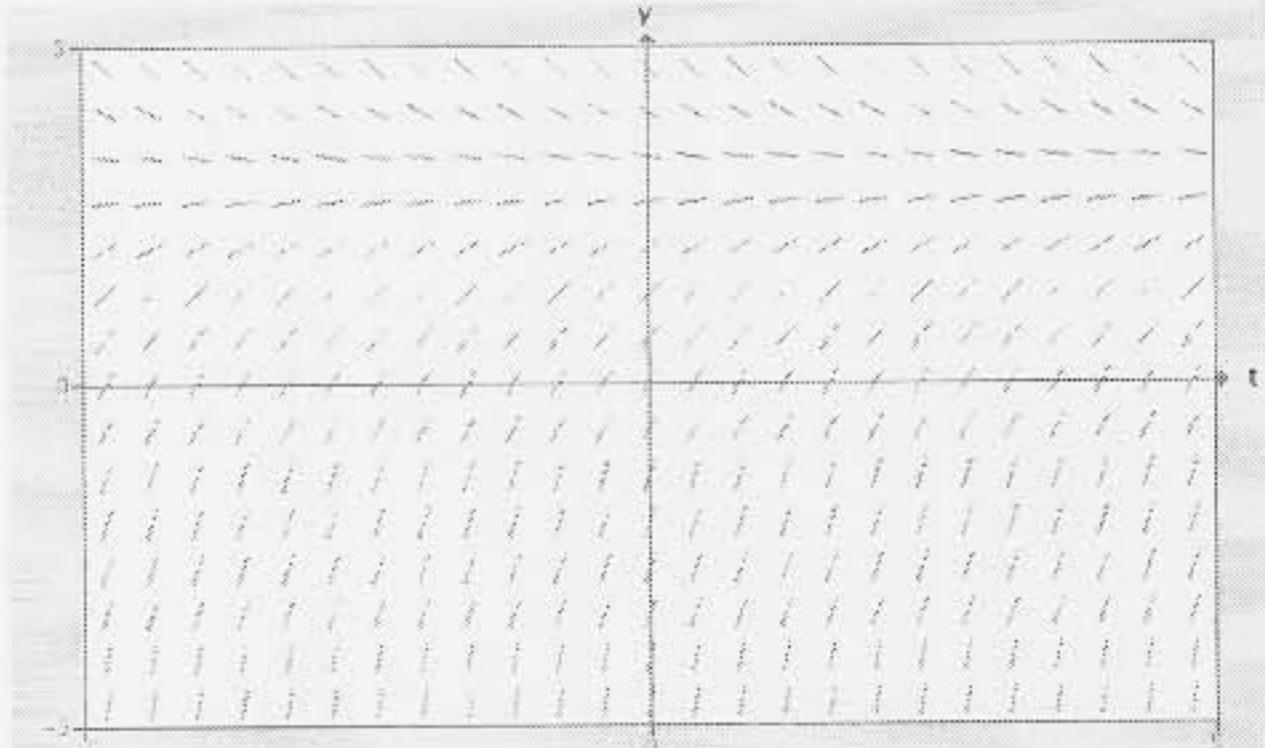
(A) (iv)



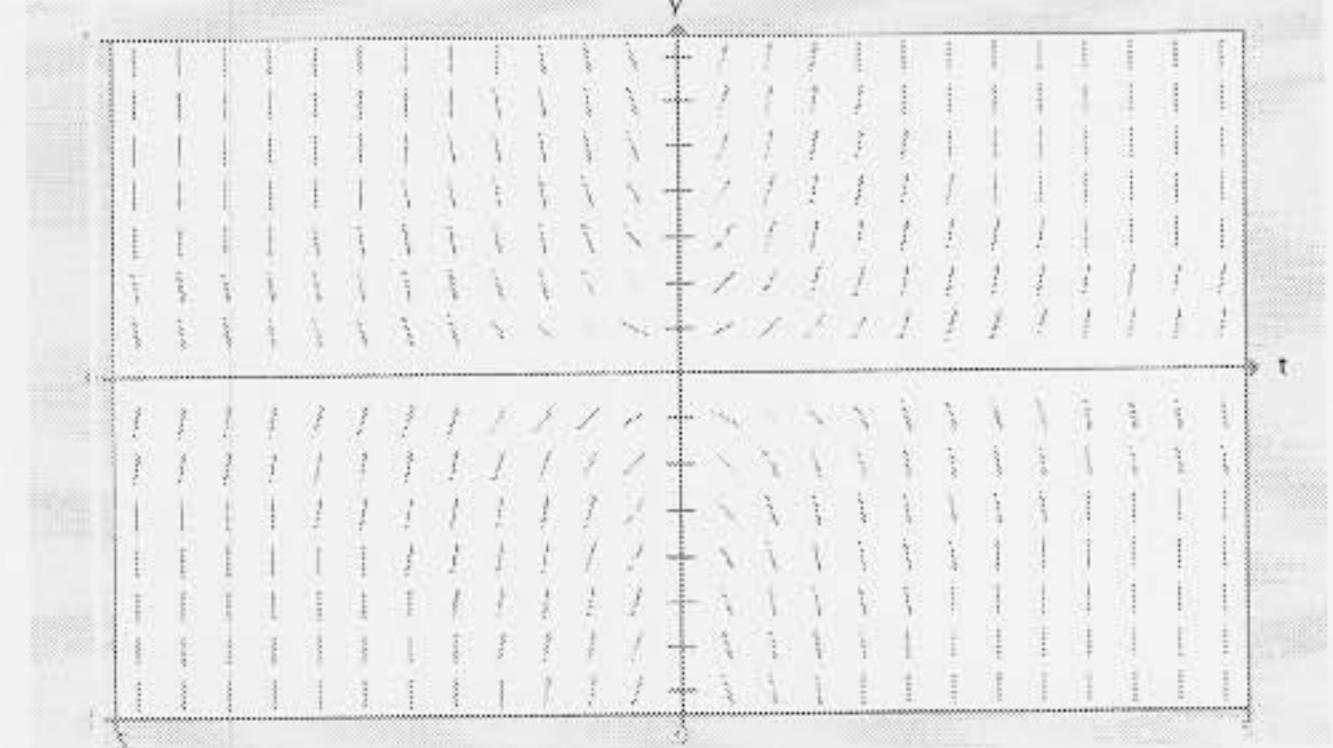
(B) (vi)



(C) (i)



(D) (v)



(A)  $\frac{dy}{dt} = f(t)$

$\frac{dy}{dt} \Big|_{t=-3} = 0$

(C)  $\frac{dy}{dt} = f(y)$

$\frac{dy}{dt} \Big|_{y=3} = 0$

(B)  $\frac{dy}{dt} = f(t, y)$

$\frac{dy}{dt} \Big|_{y=t} = 0$

(D)  $\frac{dy}{dt} = f(t, y)$

$\frac{dy}{dt} \Big|_{y=0} = 0, \quad \frac{dy}{dt} \Big|_{t=0} = 0$

2. (18 points) Solve the following differential equations or initial value problems using one of the methods from Chapter 1: Separation of Variables, Method of Undetermined Coefficients (Lucky Guess Method), or Integrating Factors. Write your solutions in explicit form.

$$(a) \frac{dy}{dt} = 3y + \cos(2t)$$

### Method of Undetermined Coefficients

$$\frac{dy}{dt} - 3y = \cos(2t)$$

$$y_h: \frac{dy_h}{dt} - 3y_h = 0$$

$$y_h = e^{rt}, \quad y_h' = re^{rt}$$

$$(r-3)e^{rt} = 0$$

$$r=3$$

$$y_h = K_1 e^{3t}$$

$$y_p = -\frac{3}{13} \cos(2t) + \frac{2}{13} \sin(2t)$$

$$y(t) = K_1 e^{3t} - \frac{3}{13} \cos(2t) + \frac{2}{13} \sin(2t)$$

$$(b) t \frac{dy}{dt} = y^2, \quad y(1) = 2$$

### Separation of Variables

$$\int \frac{1}{y^2} dy = \int \frac{1}{t} dt$$

$$\int y^{-2} dy = \int \frac{1}{t} dt$$

$$-y^{-1} = \ln|t| + C$$

$$y^{-1} = -\ln|t| - C$$

$$y = \frac{1}{-\ln|t|-C}$$

$$y(1) = \frac{1}{-\ln 1 - C} = 2, \quad C = -\frac{1}{2}$$

$$(c) y' + \left(\frac{1}{t}\right)y = 3 \cos(2t), \quad t > 0$$

$$M(t) = e^{\int \frac{1}{t} dt} = e^{\ln|t|} = t$$

### Integrating Factor Method

$$ty' + y = 3t \cos(2t)$$

$$(ty)' = 3t \cos(2t)$$

$$ty = \int 3t \cos(2t) dt$$

$$ty = \frac{3t}{2} \sin(2t) + \frac{3}{4} \cos(2t) + C$$

$$y(t) = \frac{3}{2} \sin(2t) + \frac{3}{4} \cos(2t) + \frac{C}{t}$$

$$y_p: \frac{dy_p}{dt} - 3y_p = \cos(2t)$$

$$y_p = A \cos(2t) + B \sin(2t)$$

$$y_p' = -2A \sin(2t) + 2B \cos(2t)$$

$$-2A \sin(2t) + 2B \cos(2t) - 3(A \cos(2t) + B \sin(2t)) \\ = \cos(2t)$$

$$(-2A - 3B) \sin(2t) + (2B - 3A) \cos(2t) \\ = \cos(2t)$$

$$-2A - 3B = 0$$

$$B = -\frac{2A}{3}$$

$$B = \frac{2}{13}$$

$$2B - 3A = 1$$

$$-\frac{4A}{3} - \frac{9A}{3} = 1, \quad A = -\frac{3}{13}$$

$$y(t) = \frac{1}{-t + \frac{1}{2}}$$

### Int. By Parts

$$\frac{u}{3t} + \frac{dv}{\cos(2t)}$$

$$3 \downarrow \frac{1}{2} \sin(2t)$$

$$0 \downarrow -\frac{1}{4} \cos(2t)$$

3. (16 points) Given  $\frac{dy}{dt} = y(1-y)^2$ .

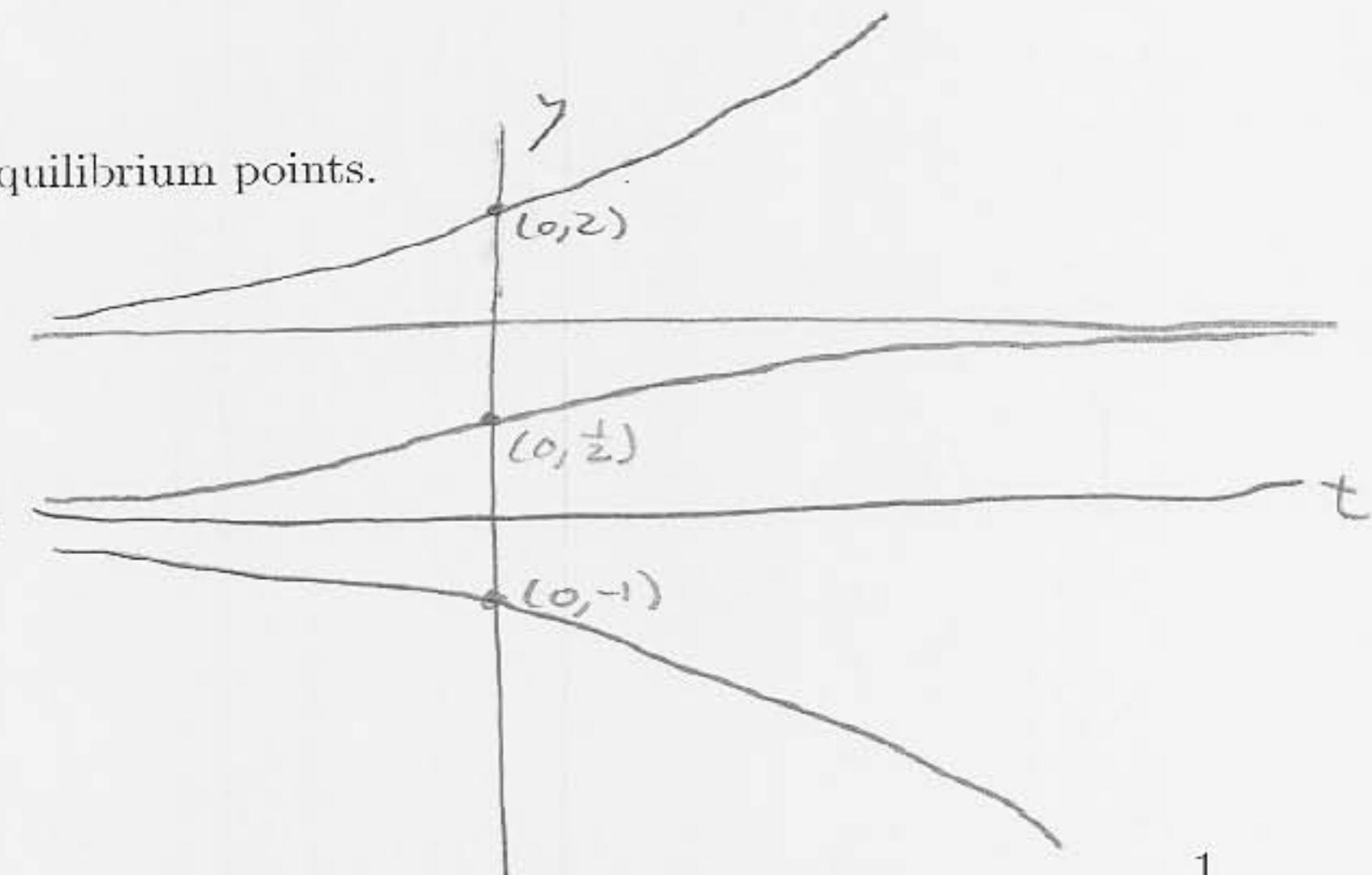
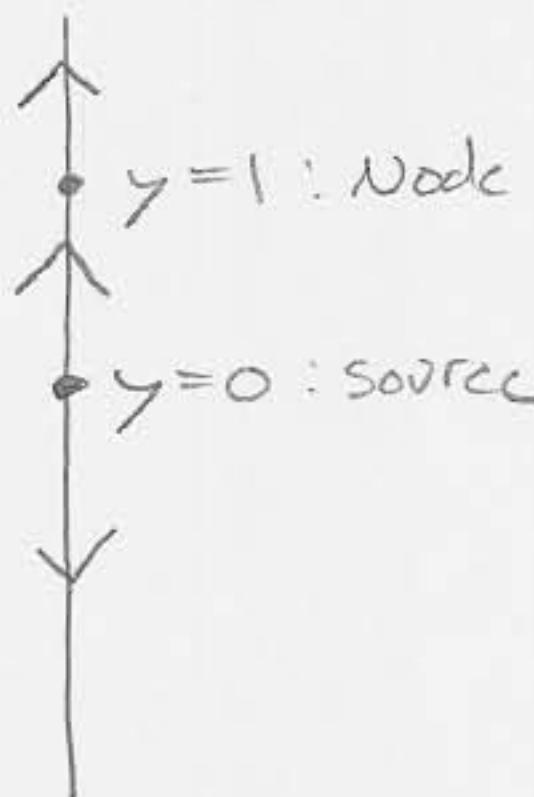
(a) Sketch the phase line and classify all of the equilibrium points.

$$y(1-y)^2 = 0 \\ \text{EP: } y=0, y=1$$

$$\frac{dy}{dt} \Big|_{y=-1} = (-1)(-1)^2 > 0$$

$$\frac{dy}{dt} \Big|_{y=\frac{1}{2}} = (\frac{1}{2})(\frac{1}{2})^2 > 0$$

$$\frac{dy}{dt} \Big|_{y=1} = 1(-1)^2 > 0$$



(b) Next to your phase line, sketch the graph of solutions satisfying the initial conditions:  $y(0) = \frac{1}{2}$ ,  $y(0) = -1$ , and  $y(0) = 2$ .

4. (16 points) Given the system of differential equations

$$\begin{aligned} \frac{dx}{dt} &= 2x \\ \frac{dy}{dt} &= x^{\frac{1}{2}} - y \end{aligned}$$

(a) Is this system linear? No

(b) Find the equilibrium solution (point) of the system.

$$\frac{dx}{dt} = 2x = 0 \\ x = 0$$

$$\frac{dy}{dt} = \boxed{x} - y = 0 \\ \underline{x=0}: -y = 0 \\ y = 0$$

(0, 0)

(c) Find the general solution of the system.

$$\frac{dx}{dt} = 2x$$

$$\frac{dx}{dt} - 2x = 0$$

$$x(t) = x_h: x_h = e^{rt}, x_h' = rc^{rt}$$

$$(r-2)c^{rt} = 0$$

$$\boxed{r=2} \\ \boxed{x(t) = k_1 e^{2t}}$$

$$\boxed{y(t) = k_2 e^{-t} + \frac{1}{2} k_1 \frac{1}{2} e^{2t}}$$

$$\frac{dy}{dt} = (k_1 e^{2t})^{\frac{1}{2}} - y$$

$$\frac{dy}{dt} + y = k_1^{\frac{1}{2}} e^t$$

$$y_h: \frac{dy_h}{dt} + y_h = 0$$

$$y_h = e^{rt}, y_h' = rc^{rt}$$

$$(r+1)c^{rt} = 0$$

$$r = -1, y_h = k_2 e^{-t}$$

$$y_p: \frac{dy_p}{dt} + y_p = k_1^{\frac{1}{2}} e^t$$

$$y_p = \alpha c^t, y_p' = \alpha c^t$$

$$\alpha c^t + \alpha c^t = k_1^{\frac{1}{2}} c^t$$

$$2\alpha = k_1^{\frac{1}{2}}, \alpha = \frac{1}{2} k_1^{\frac{1}{2}}$$

$$y_p = \frac{1}{2} k_1^{\frac{1}{2}} c^t$$

(d) Find the particular solution that satisfies  $(x(0), y(0)) = (1, 0)$ .

$$x(0) = k_1 = 1$$

$$y(0) = k_2 + \frac{1}{2} k_1^{\frac{1}{2}} = 0$$

$$k_2 = -\frac{1}{2}$$

$$\boxed{\begin{aligned} x(t) &= e^{2t} \\ y(t) &= -\frac{1}{2} e^{-t} + \frac{1}{2} e^{2t} \end{aligned}}$$

+3

3. (16 points) Given  $\frac{dy}{dt} = y(1-y)^2$ .

(a) Sketch the phase line and classify all of the equilibrium points.

(b) Next to your phase line, sketch the graph of solutions satisfying the initial conditions:  $y(0) = \frac{1}{2}$ ,  $y(0) = -1$ , and  $y(0) = 2$ .

4. (16 points) Given the system of differential equations

$$\begin{aligned}\frac{dx}{dt} &= 2x \\ \frac{dy}{dt} &= x^{\frac{1}{2}} - y\end{aligned}$$

(a) Is this system linear? NO

(b) Find the equilibrium solution (point) of the system.

$$\begin{aligned}2x &= 0 \Rightarrow x = 0 \\ \sqrt{x} - y &= 0 \\ \sqrt{x} &= y\end{aligned}$$

$$(0,0)$$

(c) Find the general solution of the system.

$$\begin{aligned}\int \frac{dx}{x} &= \int 2 dt \\ \ln|x| &= 2t + C \\ x &= C_1 e^{2t}\end{aligned}$$

$$\begin{aligned}y' + y &= (C_1 e^{2t})^{1/2} = C_1^2 e^t \\ \text{Integrating Factor} & \\ u &= e^{\int 2 dt} = e^{2t} \\ y' e^{2t} + e^{2t} y &= C_1^2 e^{2t} \\ \int (y e^{2t})' &= \int C_1^2 e^{2t} \\ y e^{2t} &= \frac{1}{2} C_1^2 e^{2t} + C_2 \\ y &= \frac{1}{2} C_1^2 e^{2t} + C_2 e^{-2t}\end{aligned}$$

OR Method of Und. Coeff  
 $y_n' + y_n = 0$   
 $\int \frac{dy_n}{y_n} = -dt$   
 $\ln|y_n| = -t + C$   
 $y_n = C_2 e^{-t}$   
 Guess:  $y_p = A e^t$   
 $\Rightarrow y_p' = A e^t$   
 $A e^t + A e^t = C_1^2 e^t$   
 $A = \frac{1}{2} C_1^2$   
 $y = C_2 e^{-t} + \frac{1}{2} C_1^2 e^t$

(d) Find the particular solution that satisfies  $(x(0), y(0)) = (1, 0)$ .

$$\begin{aligned}x(0) &= C_1 = 1 \\ y(0) &= \frac{1}{2} + C_2 = 0 \\ C_2 &= -\frac{1}{2}\end{aligned}$$

$$\begin{aligned}x &= e^{2t} \\ y &= \frac{1}{2} e^t - \frac{1}{2} e^{-t}\end{aligned}$$

5. (14 points) Given the system of differential equations

$$\begin{aligned}\frac{dx}{dt} &= x + 2y \\ \frac{dy}{dt} &= 4x + 3y\end{aligned}\quad \vec{\psi}' = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \vec{\psi}$$

Find the general solution.

$$\begin{vmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 3 - 8 = (\lambda - 5)(\lambda + 1) = 0 \Rightarrow \lambda = -1, 5$$

$$\lambda = -1: \begin{pmatrix} 1-(-1) & 2 \\ 4 & 3-(-1) \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \begin{array}{l} 2x_1 + 2y_1 = 0 \\ 4x_1 + 4y_1 = 0 \end{array} \quad \vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\lambda = 5: \begin{pmatrix} 1-5 & 2 \\ 4 & 3-5 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \begin{array}{l} -4x_2 + 2y_2 = 0 \\ 4x_2 - 2y_2 = 0 \end{array} \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\boxed{\vec{\psi} = C_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + C_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{5t}}$$

6. (18 points) Given

$$\mathbf{Y}' = \begin{pmatrix} 4 & -5 \\ 5 & -4 \end{pmatrix} \mathbf{Y}$$

Find the solution with the initial value  $\mathbf{Y}(0) = (5, 1)$ . Report your solution as one real vector.

$$\begin{vmatrix} 4-\lambda & -5 \\ 5 & -4-\lambda \end{vmatrix} = \lambda^2 - 16 + 25 = \lambda^2 + 9 = 0 \Rightarrow \lambda = \pm 3i$$

$$\lambda = 3i: \begin{pmatrix} 4-3i & -5 \\ 5 & -4-3i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \begin{array}{l} (4-3i)x - 5y = 0 \\ 5x - (4+3i)y = 0 \end{array} \quad \vec{v} = \begin{pmatrix} 4+3i \\ 5 \end{pmatrix}$$

$$\begin{aligned}e^{3i} \begin{pmatrix} 4+3i \\ 5 \end{pmatrix} &= [\cos(3t) + i\sin(3t)] \begin{pmatrix} 4+3i \\ 5 \end{pmatrix} \\ &= \begin{pmatrix} 4\cos(3t) + 4i\sin(3t) + 3i\cos(3t) - 3\sin(3t) \\ 5\cos(3t) + 5i\sin(3t) \end{pmatrix}\end{aligned}$$

$$\vec{\psi} = C_1 \begin{pmatrix} 4\cos(3t) - 3\sin(3t) \\ 5\cos(3t) \end{pmatrix} + C_2 \begin{pmatrix} 4\sin(3t) + 3\cos(3t) \\ 5\sin(3t) \end{pmatrix}$$

$$\vec{\psi}(0) = C_1 \begin{pmatrix} 4 \\ 5 \end{pmatrix} + C_2 \begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix} \quad C_1 = \frac{1}{5} \\ \frac{4}{5} + 3C_2 = 5 \Rightarrow C_2 = \frac{7}{5}$$

$$\vec{\psi} = \begin{pmatrix} 5\cos(3t) + 5\sin(3t) \\ \cos(3t) + 7\sin(3t) \end{pmatrix}$$

7. (10 points) Match each of the following eigenvalue pairs with their possible phase portrait and classify the origin for each pair, where indicated with two lines.

(i)  $\lambda_1 = i, \lambda_2 = -i$

(iv)  $\lambda_1 = -1, \lambda_2 = -4$

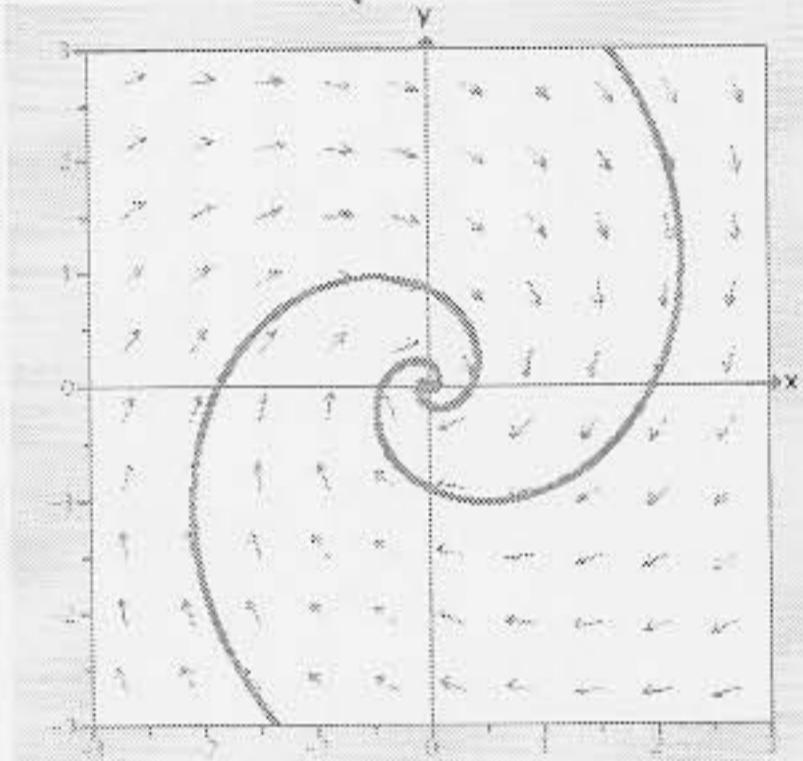
(ii)  $\lambda_1 = -1, \lambda_2 = 3$

(v)  $\lambda_1 = 0, \lambda_2 = 2$

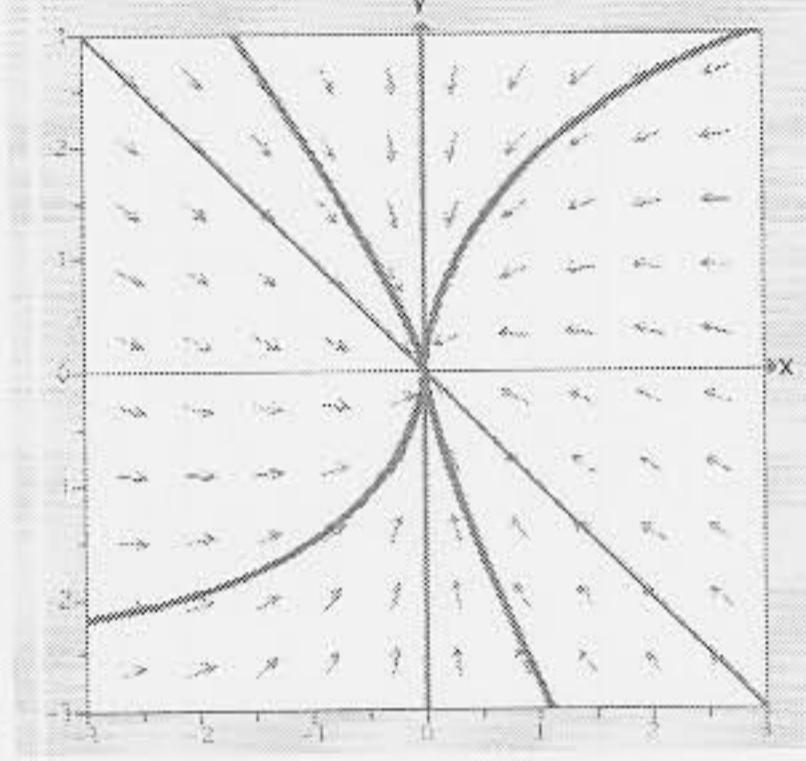
(iii)  $\lambda_1 = \lambda_2 = 2$

(vi)  $\lambda_1 = -\frac{1}{2} + i, \lambda_2 = -\frac{1}{2} - i$

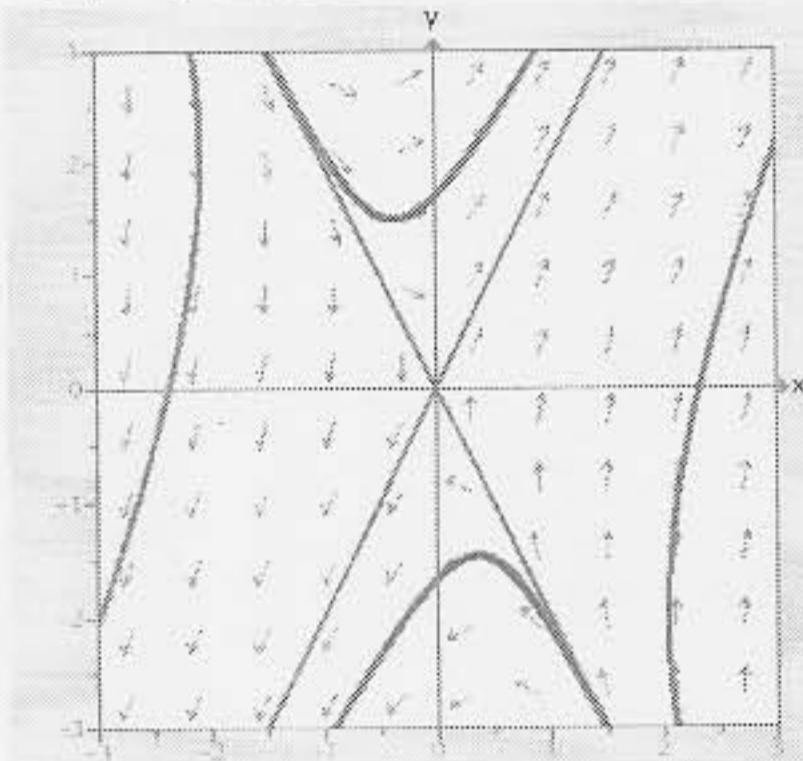
(A) (vi) Spiral sink



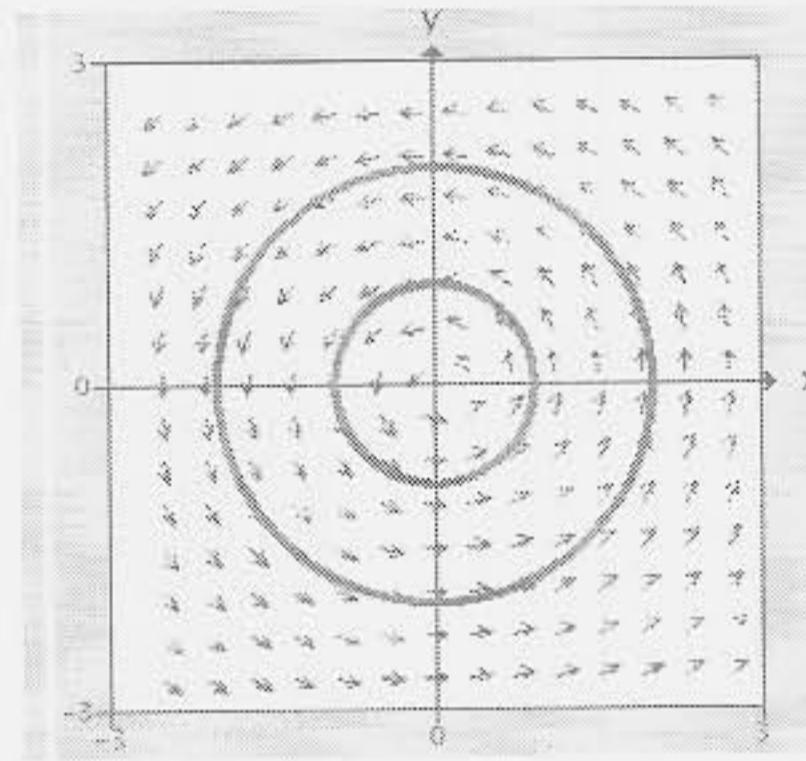
(B) (iv) Sink



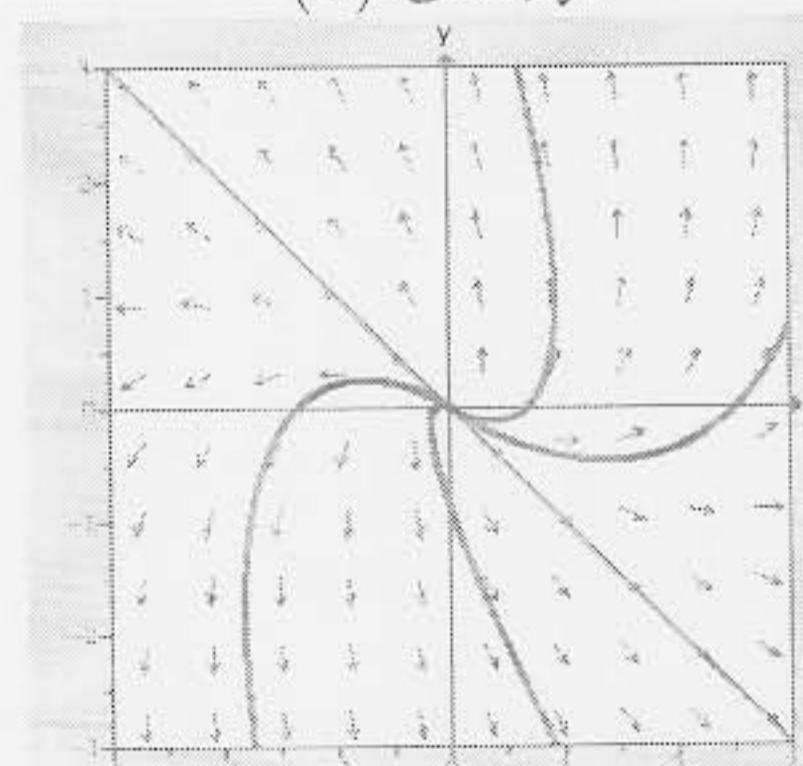
(C) (ii) Saddle



(D) (i) Center



(E) (iii)



(F) (v)

