Chapter 4

Fourier Analysis

4.1 Motivation

At the beginning of this course, we saw that superposition of functions in terms of sines and cosines was extremely useful for solving problems involving linear systems. For instance, when we studied the forced harmonic oscillator, we first solved the problem by assuming the forcing function was a sinusoid (or complex exponential). This turned out to be easy. We then argued that since the equations were linear this was enough to let us build the solution for an arbitrary forcing function if only we could represent this forcing function as a sum of sinusoids. Later, when we derived the continuum limit of the coupled spring/mass system we saw that separation of variables led us to a solution, but only **if** we could somehow represent general initial conditions as a sum of sinusoids. The representation of arbitrary functions in terms of sines and cosines is called *Fourier analysis.*

Jean Baptiste Joseph Fourier. Born: 21 March 1768 in Auxerre. Died: 16 May 1830 in Paris. Fourier trained as a priest and nearly lost his head (literally) in the French revolution. He is best known for his work on heat conduction. Fourier established the equation governing diffusion and used infinite series of trigonometric functions to solve it. Fourier was also a scientific adviser to Napoleon's army in Egypt.

4.2 The Fourier Series

So, the motivation for further study of such a Fourier superposition is clear. But there are other important reasons as well. For instance, consider the data shown in Figure 4.1.

These are borehole tiltmeter measurements. A tiltmeter is a device that measures the local tilt relative to the earth's gravitational field. The range of tilts shown here is between -40 and 40 nanoradians! (There are 2 π radians in 360 degrees, so this range corresponds to about 8 millionths of a degree.) With this sensitivity, you would expect that the dominant signal would be due to earth tides. So buried in the time-series on the top you would expect to see two dominant frequencies, one that was diurnal (1 cycle per day) and one that was semi-diurnal (2 cycles per day). If we somehow had an automatic way of representing these data as a superposition of sinusoids of various frequencies, then might we not expect these characteristic frequencies to manifest themselves in the size of the coefficients of this superposition? The answer is yes, and this is one of the principle aims of Fourier analysis. In fact, the power present in the data at each frequency is called the power spectrum. Later we will see how to estimate the power spectrum using a Fourier transform.

You'll notice in the tiltmeter spectrum that the two peaks (diurnal and semi-diurnal seem to be split; i.e., there are actually two peaks centered on 1 cycle/day and two peaks centered on 2 cycles/day. Consider the superposition of two sinusoids of nearly the same frequency:

$$\sin((\omega - \epsilon)t) + \sin((\omega + \epsilon)t).$$

Show that this is equal to

$$2\cos(\epsilon t)\sin(\omega t).$$

Interpret this result physically, keeping in mind that the way we've set the problem up, ϵ is a small number compared to ω. It might help to make some plots. Once you've figured out the interpretation of this last equation, do you see evidence of the same effect in the tiltmeter data?

There is also a *drift* in the tiltmeter. Instead of the tides fluctuating about 0 tilt, they slowly drift upwards over the course of 50 days. This is likely a drift in the instrument and not associated with any tidal effect. Think of how you might *correct* the data for this drift.

As another example Figure 4.2 shows 50 milliseconds of sound (a low C) made by a soprano saxophone and recorded on a digital oscilloscope. Next to this is the estimated power spectrum of the same sound. Notice that the peaks in the power occur at integer multiples of the frequency of the first peak (the nominal frequency of a low C).

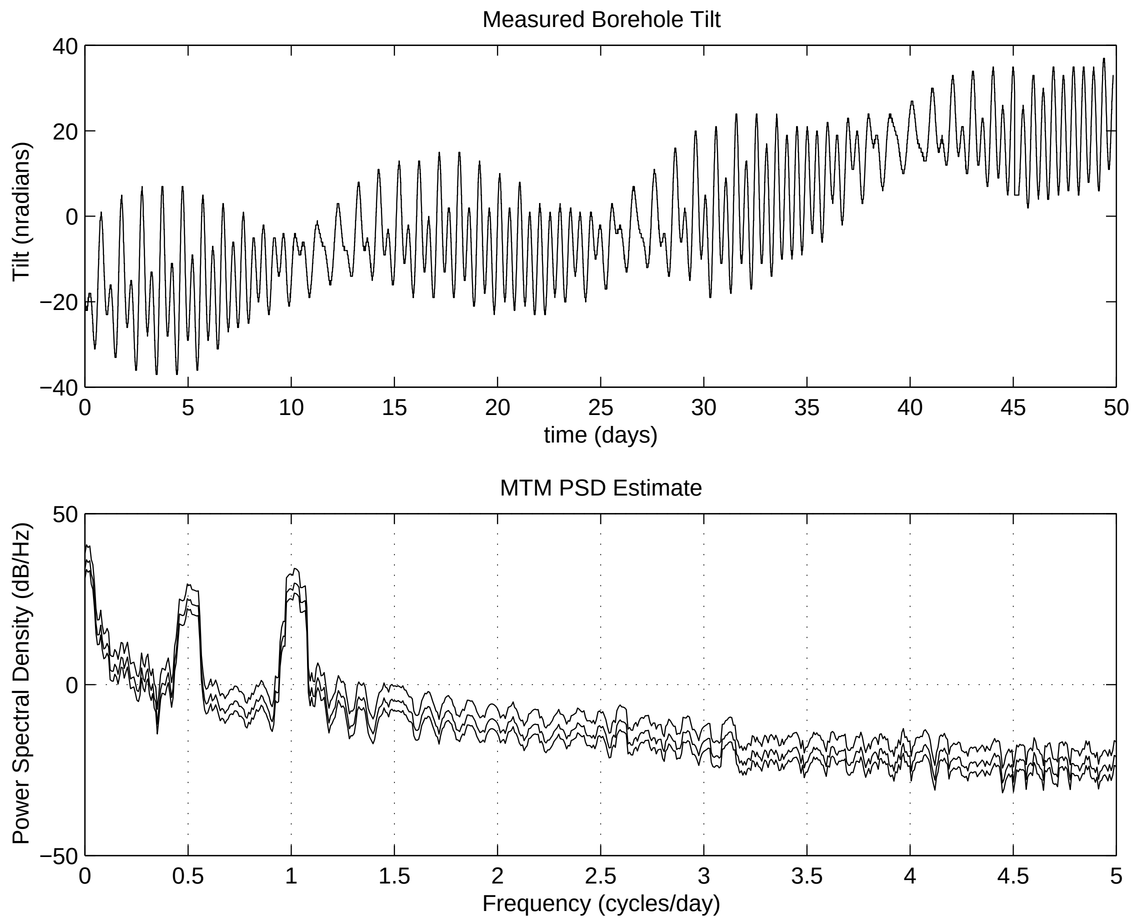

Figure 4.1: Borehole tiltmeter measurements. Data courtesy of Dr. Judah Levine (see [**?**] for more details). The plot on the top shows a 50 day time series of measurements. The figure on the bottom shows the estimated power in the data at each frequency over some range of frequencies. This is known as an estimate of the *power spectrum* of the data. Later we will learn how to compute estimates of the power spectrum of time series using the *Fourier transform*. Given what we know about the physics of tilt, we should expect that the diurnal tide (once per day) should peak at 1 cycle per day, while the semi-diurnal tide (twice per day) should peak at 2 cycles per day. This sort of analysis is one of the central goals of Fourier theory.

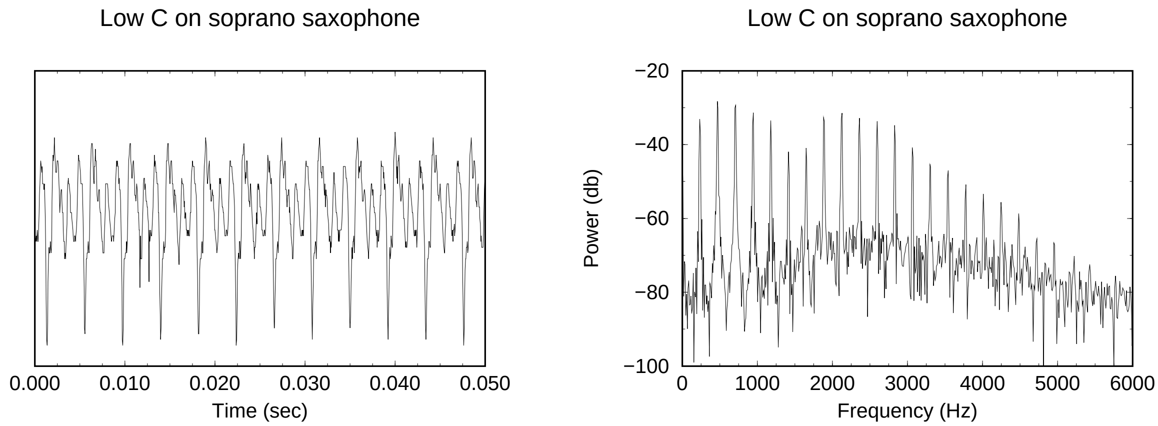

Figure 4.2: On the left is .05 seconds of someone playing low C on a soprano saxophone. On the right is the power spectrum of these data. We'll discuss later how this computation is made, but essentially what you're seeing the power as a function of frequency. The first peak on the right occurs at the nominal frequency of low C. Notice that all the higher peaks occur at integer multiples of the frequency of the first (fundamental) peak.

Definition of the Fourier Series

For a function periodic on the interval $[-l, l]$, the Fourier series is defined to be:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x/l) + b_n \sin(n\pi x/l). \tag{4.2.1}$$

or equivalently,

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/l}. \tag{4.2.2}$$

We will see shortly how to compute these coefficients. The connection between the real and complex coefficients is:

$$c_k = \frac{1}{2}(a_k - ib_k) \quad c_{-k} = \frac{1}{2}(a_k + ib_k). \tag{4.2.3}$$

In particular notice that the sine/cosine series has only positive frequencies, while the exponential series has both positive and negative. The reason is that in the former case each frequency has two functions associated with it. If we introduce a single complex function (the exponential) we avoid this by using negative frequencies. In other words, any physical vibration always involves two frequencies, one positive and one negative.

Later on you will be given two of the basic convergence theorems for Fourier series. Now let's look at some examples.

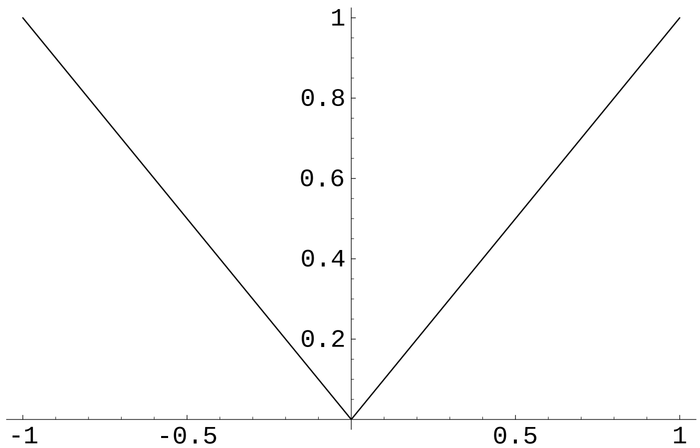

Figure 4.3: Absolute value function.

4.2.1 Examples

Let $f(x) = \text{abs}(x)$, as shown in Figure 4.3. The first few terms of the Fourier series are:

$$\frac{1}{2} - \frac{4\cos(\pi x)}{\pi^2} - \frac{4\cos(3\pi x)}{9\pi^2} - \frac{4\cos(5\pi x)}{25\pi^2} \tag{4.2.4}$$

This approximation is plotted in Figure 4.3.

Observations

Note well that the convergence is slowest at the origin, where the absolute value function is not differentiable. (At the origin, the slope changes abruptly from -1 to +1. So the left derivative and the right derivative both exist, but they are not the same.) Also, as for any even function (i.e., $f(x) = f(-x)$) only the cosine terms of the Fourier series are nonzero.

Suppose now we consider an odd function (i.e., $f(x) = -f(-x)$), such as $f(x) = x$. The first four terms of the Fourier series are

$$\frac{2\sin(\pi x)}{\pi} - \frac{\sin(2\pi x)}{\pi} + \frac{2\sin(3\pi x)}{3\pi} - \frac{\sin(4\pi x)}{2\pi} \tag{4.2.5}$$

Here you can see that only the sine terms appear, and no constant (zero-frequency) term. A plot of this approximation is shown in Figure 4.4.

So why the odd behavior at the endpoints? It's because we've assume the function is periodic on the interval $[-1, 1]$. The *periodic extension* of $f(x) = x$ must therefore have

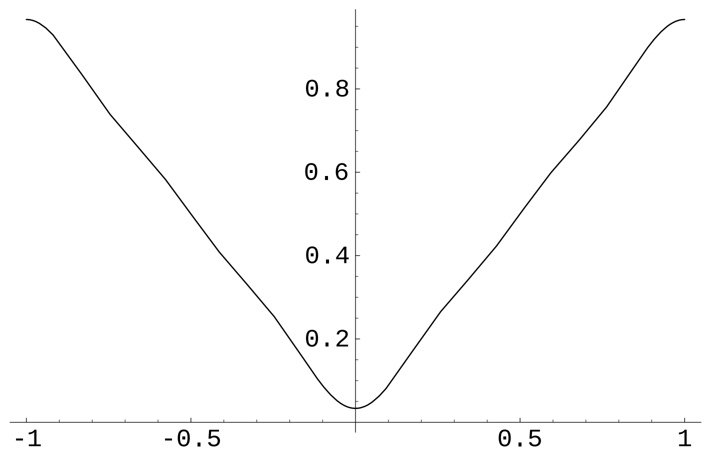

Figure 4.4: First four nonzero terms of the Fourier series of the function $f(x) = \mathrm{abs}(x)$.

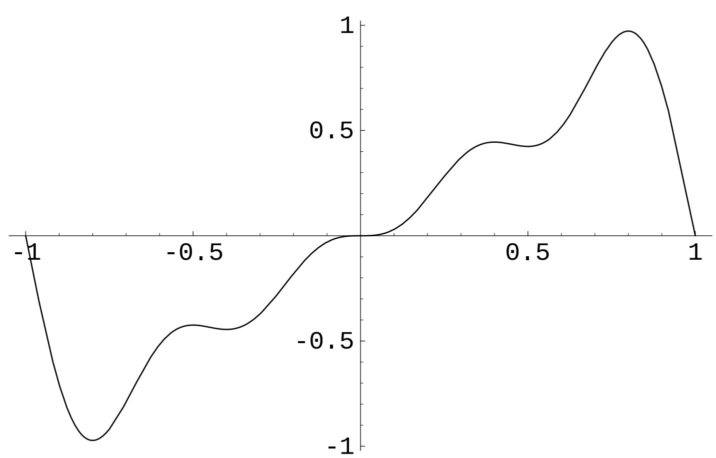

Figure 4.5: First four nonzero terms of the Fourier series of the function f(x) = x.

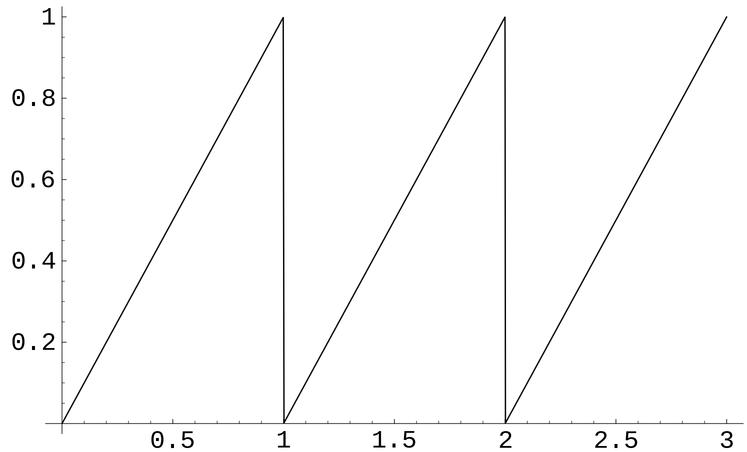

Figure 4.6: Periodic extension of the function $f(x) = x$ relative to the interval $[0, 1]$.

a sort of sawtooth appearance. In other words any non-periodic function defined on a finite interval can be used to generate a periodic function just by cloning the function over and over again. Figure 4.6 shows the periodic extension of the function $f(x) = x$ relative to the interval $[0, 1]$. It's a potentially confusing fact that the same function will give rise to different periodic extensions on different intervals. What would the periodic extension of $f(x) = x$ look like relative to the interval $[-.5, .5]$?

4.3 Superposition and orthogonal projection

Now, recall that for any set of N linearly independent vectors $\mathbf{x}_i$ in R^N, we can represent an arbitrary vector $\mathbf{z}$ in R^N as a superposition

$$\mathbf{z} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_N\mathbf{x}_N, \tag{4.3.1}$$

which is equivalent to the linear system

$$\mathbf{z} = X \cdot \mathbf{c} \tag{4.3.2}$$

where X is the matrix whose columns are the $\mathbf{x}_i$ vectors and $\mathbf{c}$ is the vector of unknown expansion coefficients. As you well know, matrix equation has a unique solution $\mathbf{c}$ if and only if the $\mathbf{x}_i$ are linearly independent. But the solution is especially simple if the $\mathbf{x}_i$ are orthogonal. Suppose we are trying to find the coefficients of

$$\mathbf{z} = c_1\mathbf{q}_1 + c_2\mathbf{q}_2 + \cdots + \mathbf{q}_N, \tag{4.3.3}$$

when $\mathbf{q}_i \cdot \mathbf{q}_j = \delta_{ij}$. In this case we can find the coefficients easily by projecting onto the orthogonal directions:

$$c_i = \mathbf{q}_i \cdot \mathbf{z}, \tag{4.3.4}$$

or, in the more general case where the $\mathbf{q}$ vectors are orthogonal but not necessarily normalized

$$c_i = \frac{\mathbf{q}_i \cdot \mathbf{z}}{\mathbf{q}_i \cdot \mathbf{q}_i}. \tag{4.3.5}$$

We have emphasized throughout this course that functions are vectors too, they just happen to live in an infinite dimensional vector space (for instance, the space of square integrable functions). So it should come as no surprise that we would want to consider a formula just like 4.3.3, but with functions instead of finite dimensional vectors; e.g.,

$$f(x) = c_1 q_1(x) + c_2 q_2(x) + \cdots + c_n q_n(x) + \cdots. \tag{4.3.6}$$

In general, the sum will require an infinite number of coefficients c_i, since a function has an infinite amount of information. (Think of representing $f(x)$ by its value at each point x in some interval.) Equation 4.3.6 is nothing other than a Fourier series if the $q(x)$ happen to be sinusoids. Of course, you can easily think of functions for which all but a finite number of the coefficients will be zero; for instance, the sum of a finite number of sinusoids.

Now you know exactly what is coming. If the basis functions $q_i(x)$ are "orthogonal", then we should be able to compute the Fourier coefficients by simply projecting the function $f(x)$ onto each of the orthogonal "vectors" $q_i(x)$. So, let us define a dot (or inner) product for functions on an interval $[-l, l]$ (this could be an infinite interval)

$$(u, v) \equiv \int_{-l}^{l} u(x) v(x) dx. \tag{4.3.7}$$

Then we will say that two functions are orthogonal if their inner product is zero.

Now we simply need to show that the sines and cosines (or complex exponentials) are orthogonal. Here is the theorem. Let $\phi_k(x) = \sin(k\pi x/l)$ and $\psi_k(x) = \cos(k\pi x/l)$. Then

$$(\phi_i, \phi_j) = (\psi_i, \psi_j) = l\delta_{ij} \tag{4.3.8}$$

$$(\phi_i, \psi_j) = 0. \tag{4.3.9}$$

The proof, which is left as an exercise, makes use of the addition formulae for sines and cosines. (If you get stuck, the proof can be found in [2], Chapter 10.) A similar result holds for the complex exponential, where we define the basis functions as $\xi_k(x) = e^{ik\pi x/l}$.

Using Equations 4.3.8 and 4.3.9 we can compute the Fourier coefficients by simply projecting $f(x)$ onto each orthogonal basis vector:

$$a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos(n\pi x/l) dx = \frac{1}{l}(f, \psi_n), \tag{4.3.10}$$

and

$$b_n = \frac{1}{l} \int_{-l}^{l} f(x) \sin(n\pi x/l) dx = \frac{1}{l}(f, \phi_n). \tag{4.3.11}$$