MATH 348 - Advanced Engineering Mathematics November 18, 2009

Homework 9, Fall 2009 **Due**: November 30, 2009

Review of Ordinary Differential Equations - Introduction to Partial Differential Equations

1. Given,

$$ay'' + by' + cy = f(x). \tag{1}$$

(a) Find all solutions to the homogeneous version of (1). List this information in a table based highlighting how the general solution changes with the discriminant $D = b^2 - 4ac$. [1]

(b) Fill in the following table, which outlines the choices you would make for the particular solution for various choices of $f(x)$:[2]

$f(x)$	$y_p(x)$
x^4	
$\cos(\beta x)$	
$e^{\alpha x} + \sin(x) + x$	
$e^{\alpha x}\sin(\beta x)$	
$x^2 e^{\alpha x}\cos(\beta x)$	

(c) Suppose that $a = 1$, $b = 0$, $c = 9$ and $f(x) = \cos(3x)$. Find the general solution to (1).[3]

2. Consider the boundary value problem:

$$\begin{aligned} y'' + \lambda y &= 0 && (2)\\ y'(0) = y'(L) &= 0, \quad L \in \mathbb{R}^+, && (3) \end{aligned}$$

(a) Assuming that $\lambda \in \mathbb{R}$, find the general solution to (2).

(b) Apply the boundary conditions (3) to the solutions found in (a) and calculate the possible values of λ such that these solutions also satisfy the boundary conditions (3).

Hint: For problem (a) use the assumption that $y(x) = e^{rx}$, $r \in \mathbb{R}$ and find solutions for the three cases $\lambda < 0$, $\lambda > 0$, $\lambda = 0$. Concerning (b), this time, two of the three solution types will non-trivially satisfy (2)-(3).

[1] First assume that $y(x) = e^{rx}$ to find the characteristic equation $ar^2 + br + c = 0$. After this find the roots of the characteristic equation and thus the general solutions to (1). Note that these solutions will depend on the discriminant of the characteristic equation.

[2] Do not solve for the undetermined coefficients. Also, you may assume that a, b, c are such that no part of the homogeneous solution matches your particular solutions.

[3] Keep in mind that this is a resonant case and the choice for particular solution will need a coefficient of x. Don't forget to use the product rule!

3. Consider the ordinary differential equation:

$$y'' - y = 0 \tag{4}$$

We know that the general solution to this equation is $y(x) = c_1 e^x + c_2 e^{-x}$. It is common to write the solutions to (4) in terms of the hyperbolic trigonometric functions, $\sinh(x) = \dfrac{e^x - e^{-x}}{2}$, $\cosh(x) = \dfrac{e^x + e^{-x}}{2}$. [4]

   (a) Show that $y(x) = b_1 \sinh(x) + b_2 \cosh(x)$ is a solution to the differential equation (3).

   (b) Show that if $c_1 = \dfrac{b_1 + b_2}{2}$ and $c_2 = \dfrac{b_1 - b_2}{2}$ then $y(x) = c_1 e^x + c_2 e^{-x} = b_1 \cosh(x) + b_2 \sinh(x)$.

   (c) Assume that $y(x) = \displaystyle\sum_{n=0}^{\infty} a_n x^n$ to find the general solution of (3) in terms of the hyperbolic sine and cosine functions. [5]

4. In class we will study the heat and wave equation in one spatial dimension. Though these problems are considered rather plain they exist as the basic components for the study of flows and vibrations. Consider the following reference material,

   - `http://en.wikipedia.org/wiki/Laplace's_equation`
   - `http://en.wikipedia.org/wiki/Heat_equation`
   - `http://en.wikipedia.org/wiki/Diffusion_equation`
   - `http://en.wikipedia.org/wiki/Wave_equation`
   - `http://en.wikipedia.org/wiki/Dispersion_(water_waves)`

   and respond to the questions.

   (a) Laplace's equation is ubiquitous and is lurking in the shadows of everything we study through this chapter. Solutions to Laplace's equation are called Harmonic functions and have the property of being *analytic*. What does this term mean?

   (b) For the heat equation u represents the temperature of the medium as a function of space and time. It can be shown that u obeys the maximum principle. Explain the physical interpretation of this principle. [6]

   (c) Explain the relationship between the heat equation and the diffusion equation as given by the websites. What is the fundamental principle used to derive the diffusion equation? [7]

   (d) What physical phenomenon modeled by the linear wave equation? What about the nonlinear wave equation? [8]

   (e) Define dispersion and give a physical example of dispersion in travailing waves.

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[4] After you finish this problem you may want to look back at the previous problem and see if you can reformulate its solutions in terms of hyperbolic trigonometric functions. This will cause the $\lambda > 0$ case to look similar to the $\lambda < 0$ case, which is a primary reason for using these functions.

[5] The hyperbolic sine and cosine have the following Taylor's series representations centred about $x = 0$,

$$\cosh(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \qquad \sinh(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}. \tag{5}$$

It is worth noting that these are basically the same Taylor series as cosine/sine with the exception that the signs of the terms do not alternate. From this we can gather a final connection for wrapping all of these functions together. If you have the Taylor series for the exponential function and extract the even terms from it then you have the hyperbolic cosine function. Taking the hyperbolic cosine function and alternating the sign of its terms gives you the cosine function. Extracting the odd terms from the exponential function gives the same statements for the hyperbolic sine and sine functions. The reason these functions are connected via the imaginary number system is because when i is raised to integer powers it will produce these exact sign alternations. So, if you remember $e^x = \sum_{n=0}^{\infty} x^n/n!$ and $i = \sqrt{-1}$ then the rest (hyperbolic and non-hyperbolic trigonometric functions) follows!

[6] This relates to the derivations conducted in class and the point here is that most PDE manifest from some sort of conservation principle. So, if the PDE is unbelievable then at least we can put our hopes on the statement, 'energy can be neither destroyed nor created.' I know it helps me sleep at night.

[7] Short answer: They are the same beast all that changes is what the parameters and unknown function are speaking about physically.

[8] This is discussed in the introduction. Linear equations tend to be the most pertinent but they are also the most irrelevant. Well, not really but the basic idea is that linear phenomenon is common and well understood. However, like most linear problems, they exist as an idealization of more complicated nonlinear phenomenon and highlights the classic mathematical trick of replacing bad quantities (nonlinear objects) with good quantities (linear object) at the sacrifice of analytic precision.

5. Show that the following functions are solutions to their corresponding PDE's.

   (a) $u(x,t) = f(x-ct) + g(x+ct)$ for the 1-D wave equation.

   (b) $u(x,t) = e^{-4\omega^2 t}\sin(\omega x)$ where $c = 2$ for the 1-D heat equation.

   (c) $u(x,y) = x^4 + y^4$ where $f(x,y) = 12(x^2+y^2)$ for the 2-D Poisson equation.

   (d) $u(x,y,z) = \dfrac{1}{\sqrt{x^2+y^2+z^2}}$ for the 3-D Laplace equation.

Note: The PDE in question are,

- Laplace's Equation : $\triangle u = 0$
- Poisson's Equation : $\triangle u = f(x,y,z)$
- Heat/Diffusion Equation : $u_t = c^2 \triangle u$
- Wave Equation : $u_{tt} = c^2 \triangle u$

and can be found on page 563 of Kreyszig. The following will outline some common notations. It is assumed that all differential operators are being expressed in Cartesian coordinates. [9]

- Notations for partial derivatives

$$\frac{\partial u}{\partial x} = u_x = \partial_x u \tag{6}$$

- Nabla the differential operator

$$\nabla = \begin{bmatrix} \partial_x \\ \partial_y \\ \partial_z \end{bmatrix} \tag{7}$$

- Gradient of a scalar function

$$\nabla u = \begin{bmatrix} \partial_x u \\ \partial_y u \\ \partial_z u \end{bmatrix} \tag{8}$$

- Divergence of a vector

$$\nabla \cdot \mathbf{v} = v_x + v_y + v_z \tag{9}$$

- Curl of a vector

$$\nabla \times \mathbf{v} = \begin{bmatrix} \partial_y v_3 - \partial_z v_2 \\ \partial_z v_1 - \partial_x v_3 \\ \partial_x v_2 - \partial_y v_1 \end{bmatrix} \tag{10}$$

- Notations for the Laplacian

$$\triangle u = \nabla \cdot \nabla u = \begin{bmatrix} \partial_x \\ \partial_y \\ \partial_z \end{bmatrix} \cdot \begin{bmatrix} \partial_x u \\ \partial_y u \\ \partial_z u \end{bmatrix} \tag{11}$$

$$= \partial_{xx} u + \partial_{yy} u + \partial_{zz} u \tag{12}$$

$$= u_{xx} + u_{yy} + u_{zz} \tag{13}$$

$$= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \tag{14}$$

[9] Of course others have worked out the common coordinate systems, which requires some elbow grease and the multivariate chain rule. Those interested in the results can find them at `http://en.wikipedia.org/wiki/Del_in_cylindrical_and_spherical_coordinates`