2.3.4 (8 pts) Use set builder notation to specify the following sets:

(a) The set of all integers greater than or equal to 5. $\{x \in \mathbb{Z} \,|\, x \geq 5\}$

(b) The set of all even integers. $\{x \in \mathbb{Z} \,|\, x = 2a, a \in \mathbb{Z}\}$

(c) The set of all positive rational numbers. $\mathbb{Q}^+$ or $\{x \in \mathbb{Q} \,|\, x > 0\}$

(d) The set of all real numbers greater than 1 and less than 7. $\{x \in \mathbb{R} \,|\, 1 < x < 7\}$

2.3.5 (6 pts) For each of the following sets, use English to describe the set or use the roster method to specify all of the elements of the set.

(a) $\{x \in \mathbb{R} \,|\, -3 \leq x \leq 5\}$
The set of all real numbers between -3 and 5, inclusive.

(b) $\{x \in \mathbb{Z} \,|\, -3 \leq x \leq 5\}$
The set of all integers between -3 and 5, inclusive. or
The set containing -3, -2, -1, 0, 1, 2, 3, 4, 5.

(c) $\{x \in \mathbb{R} \,|\, x^2 = 16\}$
The set containing 4 and -4.

2.4.3 (12 pts) Assume the universal set for each variable is the set of integers. Write each of the following statements as an English sentence that does not use the symbols for quantifiers.

(a) $(\exists m)(\exists n)(m > n)$
There exists an m such that there is an n where $m > n$.

(b) $(\exists m)(\forall n)(m > n)$
There exists an m such that for every n, $m > n$.

(c) $(\forall m)(\exists n)(m > n)$
For each m, there is an n such that $m > n$.

(d) $(\forall m)(\forall n)(m > n)$
For every m, m is greater than every n.

(e) $(\exists n)(\forall m)(m^2 > n)$
There exists an n such that the square of every m is greater.

(f) $(\forall n)(\exists m)(m^2 > n)$
For every n, there exists an m such that $m^2 > n$.

3.1.8 (4 pts) Determine if each of the following propositions is true or false. Justify each conclusion.

(a) For all integers a and b, if $ab \equiv 0 \pmod 6$, then $a \equiv 0 \pmod 6$ or $b \equiv 0 \pmod 6$.
This proposition is false. Consider $a = 2$ and $b = 3$.

$$2 \cdot 3 \equiv_6 0 \text{ however } 2 \not\equiv_6 0 \text{ and } 3 \not\equiv_6 0$$

(b) For all $a \in \mathbb{Z}$, if $a^2 \equiv 4 \pmod 8$, then $a \equiv 2 \pmod 8$. This proposition is false. Consider $a = 6$.

$$36 \equiv_8 4 \text{ however } 6 \not\equiv_8 2$$

3.1.11 (10 pts) Let r be a positive real number. The equation for a circle of a radius r whose center is the origin is $x^2 + y^2 = 1$.

(a) +3 Use implicit differentiation to determine $\frac{dy}{dx}$

$$\frac{dy}{dx} = -\frac{x}{y}$$

(b) +2 Let (a, b) be a point on the circle with $a \neq 0$ and $b \neq 0$. Determine the slope of the line tangent to the circle at the point (a, b).

From part (a), the slope of the line tangent to the circle at (a, b) is $-\frac{a}{b}$.

(c) +5 Prove that the radius of the circle to the point (a, b) is perpendicular to the line tangent to the circle at the point (a, b).

Proof. Two lines are perpendicular if and only if their corresponding slopes are negative reciprocals. From part (b), we see that the slope of the tangent line at the point (a, b) is given by $m_1 = -\frac{a}{b}$. The slope of the radius from the center to (a, b) is given by $m_2 = \frac{b-0}{a-0} = \frac{b}{a}$. From this we see that $m_1 \cdot m_2 = -1$ and thus the slopes are negative reciprocals. Therefore, the radius of the circle to the point (a, b) is perpendicular to the line tangent to the circel at (a, b). □

3.1.12 (12 pts) Determine if each of the following statements is true or false. Provide a counterexample for statements that are false and provide a complete proof for those that are true.

(a) +2 For all real numbers x and y, $\sqrt{xy} \leq \frac{x+y}{2}$.

This proposition is false. Consider $x = -1$ and $y = -4$

$$\sqrt{-1 \cdot -2} = \sqrt{4} = 2 \not\leq \frac{-1 + -4}{2} = -\frac{5}{2}$$

(b) +5 For all real numbers x and y, $xy \leq \left(\frac{x+y}{2}\right)^2$.

This proposition is true.

Proof. Let $x, y \in \mathbb{R}$. Then

$$\begin{aligned}
(x-y)^2 \geq 0 &\Rightarrow x^2 - 2xy + y^2 \geq 0 \\
&\Rightarrow x^2 + 2xy + y^2 \geq 4xy \\
&\Rightarrow (x+y)^2 \geq 4xy \\
&\Rightarrow \frac{(x+y)^2}{4} \geq xy \\
&\Rightarrow \left(\frac{x+y}{2}\right)^2 \geq xy
\end{aligned}$$

Thus, for all real numbers x and y, $xy \leq \left(\frac{x+y}{2}\right)^2$. □

(c) +5 For all nonnegative real numbers x and y, $\sqrt{xy} \leq \frac{x+y}{2}$.
This proposition is true.

Proof. We begin by duplicating the steps used in part(b),
Let $x, y \in \mathbb{R}$ with $x, y \geq 0$. Then

$$\begin{aligned}(x-y)^2 \geq 0 &\Rightarrow x^2 - 2xy + y^2 \geq 0 \\ &\Rightarrow x^2 + 2xy + y^2 \geq 4xy \\ &\Rightarrow (x+y)^2 \geq 4xy \\ &\Rightarrow \frac{(x+y)^2}{4} \geq xy \\ &\Rightarrow \left(\frac{x+y}{2}\right)^2 \geq xy\end{aligned}$$

Since for all $x \geq 0$, $\sqrt{x^2} = x$ and since for $x, y \geq 0, xy \geq 0$, we can take the square root of both sides, maintaining the inequality relationship.

$$\Rightarrow \frac{x+y}{2} \geq \sqrt{xy}$$

Thus, for all nonnegative real numbers x and y, $\sqrt{xy} \leq \frac{x+y}{2}$. □

3.1.13 (5 pts) Use one of the true inequalities in Exercise (12) to prove the following proposition.

For each real number, a, the value of x that gives the maximum value of $x(a-x)$ is $x = \frac{a}{2}$.

Proof. Using part (b) from above with $x = x$ and $y = (a-x)$, we have

$$x(a-x) \leq \frac{x+(a-x)}{2} = \frac{a}{2}$$

Thus, $\frac{a}{2}$ is the maximum value of $x(a-x)$. Solving for x, gives

$$x(a-x) = \frac{a}{2} \Rightarrow x^2 - ax + \frac{a}{2} = 0 \Rightarrow \left(x - \frac{a}{2}\right)^2 = 0 \Rightarrow x = \frac{a}{2}$$

Therefore, $x = \frac{a}{2}$ gives the maximum value of $x(a-x)$. □

3.2.10 (5 pts) Prove that for each integer a, if $a^2 - 1$ is even, then 4 divides $a^2 - 1$.

Proof. Let $a \in \mathbb{Z}$ such that $a^2 - 1$ is even. Then there exists a $k \in \mathbb{Z}$ such that
$a^2 - 1 = 2k \Rightarrow a^2 = 2k + 1$. Thus, a^2 and therefore a are odd.
Then there exists a $j \in Z$ such that $a = 2j + 1$. Now consider,

$$a^2 - 1 = (2j+1)^2 - 1 = 4j^2 + 4j + 1 - 1 = 4(j^2 + j)$$

Therefore, 4 divides $a^2 - 1$. □

3.2.11 (5 pts) Prove that for all integers a and m, if a and m are the lengths of the sides of a right triangle and $m+1$ is the length of the hypotenuse, then a is an odd integer.

Proof. Let $a, m \in \mathbb{Z}$ defined as given above. Then by the Pythagorean Theorem,

$$a^2 + m^2 = (m+1)^2 \Rightarrow a^2 + m^2 = m^2 + 2m + 1 \Rightarrow a^2 = 2m + 1$$

Then, a^2 is odd and thus a is odd. □

3.3.16 (8 pts) Three natural numbers a, b, and c with $a < b < c$ are called a **Pythagorean triple** provided that $a^2 + b^2 = c^2$. For example, the numbers 3, 4, and 5 form a Pythagorean triple, and the numbers 5, 12, and 13 form a Pythagorean triple.

(a) +1 Verify that if $a = 20$, $b = 21$, and $c = 29$, then $a^2 + b^2 = c^2$, and hence, 20, 21, and 29 form a Pythagorean triple.
I trust that everyone can use the Pythagorean Theorem.

(b) +2 Determine two other Pythagorean triples. That is, find integers a, b and c such that $a^2 + b^2 = c^2$.
I trust that everyone can use the Pythagorean Theorem.

(c) +5 Is the following proposition true or false? Justify your conclusion.

Let a, b, and c be integers. If $a^2 + b^2 = c^2$, then a is even or b is even.

The proposition is true.

Proof. Using a proof by contradiction, let $a, b, c \in \mathbb{Z}$ such that $a^2 + b^2 = c^2$ and assume that both a and b are odd. Then there exists $m, n \in \mathbb{Z}$ such that

$$a = 2m + 1 \text{ and } b = 2n + 1$$

Then

$$\begin{aligned} a^2 + b^2 = c^2 &\Rightarrow (2m+1)^2 + (2n+1)^2 = c^2 \\ &\Rightarrow 4m^2 + 4m + 4n^2 + 4n + 2 = c^2 \\ &\Rightarrow 4k + 2 = c^2, \text{ for } k = m^2 + m + n^2 + n \\ &\Rightarrow 2(2k+1) = c^2 \end{aligned}$$

We see that our perfect square c must be the product of an odd and even integer - an impossibility and thus a contradiction to our assumption that both a and b are odd[1]. □

[1] You could also carry the proof further as follows

$$2(2k+1) = c^2 \Rightarrow c^2 \text{ is even } \Rightarrow c \text{ is even } \Rightarrow c = 2j \Rightarrow 2(2k+1) = 4j^2 \Rightarrow 2k+1 = 2j^2$$

This implies that $2k + 1$ is divisible by 2, also a contradiction.