Homework #2 Solutions:

1. See second file.

2. Let $A = \begin{bmatrix} 1 & -3 & -4 \\ -3 & 2 & 6 \\ 5 & -1 & -8 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$. Show that the equation $A\mathbf{x} = \mathbf{b}$ does not have a solution for all possible $\mathbf{b}$ and describe the set of all $\mathbf{b}$ for which $A\mathbf{x} = \mathbf{b}$ *does* have a solution.
Row reducing the corresponding augmented matrix yields:

$$\begin{bmatrix} 1 & -3 & -4 & b_1 \\ -3 & 2 & 6 & b_2 \\ 5 & -1 & -8 & b_3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -3 & -4 & b_1 \\ 0 & -7 & -6 & b_2 + 3b_1 \\ 0 & 14 & 12 & b_3 - 5b_1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -3 & -4 & b_1 \\ 0 & -7 & -6 & b_2 + 3b_1 \\ 0 & 0 & 0 & b_1 + 2b_2 + b_3 \end{bmatrix}$$

If $b_1 + 2b_2 + b_3 \neq 0$ the system is inconsistent. However, if $b_1 + 2b_2 + b_3 = 0 \Rightarrow b_1 = -2b_2 - b_3$, the system will be consistent. Thus, for all $\mathbf{b} = b_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + b_3 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ the system is consistent.

3. Define

$$A = \begin{bmatrix} 1 & 3 & -2 & 2 \\ 0 & 1 & 1 & -5 \\ 1 & 2 & -3 & 7 \\ -2 & -8 & 2 & -1 \end{bmatrix}, \ B = \begin{bmatrix} 5 & 3 \\ -4 & 7 \\ 9 & -2 \end{bmatrix}, \text{ and } b = \begin{bmatrix} 22 \\ 20 \\ 15 \end{bmatrix}.$$

(a) Do the columns of A span $\mathbb{R}^4$?
Row reducing A to echelon form yields:

$$\rightarrow \begin{bmatrix} 1 & 3 & -2 & 2 \\ 0 & 1 & 1 & -5 \\ 0 & -1 & -1 & 5 \\ 0 & -2 & -2 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & -2 & 2 \\ 0 & 1 & 1 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -7 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & -2 & 2 \\ 0 & 1 & 1 & -5 \\ 0 & 0 & 0 & -7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since there is not a pivot in every row, A does not span $\mathbb{R}^4$.

(b) Is $\mathbf{b}$ a linear combination of the columns of B?
Row reducing the corresponding augmented matrix yields:

$$\begin{bmatrix} 5 & 3 & 22 \\ -4 & 7 & 20 \\ 9 & -2 & 15 \end{bmatrix} \rightarrow \begin{bmatrix} 5 & 3 & 22 \\ 0 & 47/5 & 188/5 \\ 0 & -37/5 & -123/5 \end{bmatrix} \rightarrow \begin{bmatrix} 5 & 3 & 22 \\ 0 & 47 & 188 \\ 0 & -37 & -123 \end{bmatrix} \rightarrow \begin{bmatrix} 5 & 3 & 22 \\ 0 & 47 & 188 \\ 0 & 0 & 1175/47 \end{bmatrix}$$

Since the last column is a pivot column, this system is inconsistent. Thus $\mathbf{b}$ can not be written as a linear combination of the columns of B.

4. Let $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 9 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 3 \\ -4 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix}$.
Forming the corresponding matrix and row-reducing to echelon form:

$$\begin{bmatrix} 0 & 3 & -4 \\ 9 & -4 & 1 \\ 1 & 1 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & 1 \\ 9 & -4 & 1 \\ 0 & 3 & -4 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & -13 & -8 \\ 0 & 3 & -4 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & -13 & -8 \\ 0 & 0 & -76/13 \end{bmatrix}$$

(a) Does $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ form a linearly independent set?
Since there is a pivot in each column, the set is linearly independent.

(b) Does $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ span $\mathbb{R}^3$?

Since there is a pivot in every row, the set spans $\mathbb{R}^3$

5. Do the columns of $A = \begin{bmatrix} -4 & -3 & 0 \\ 0 & -1 & 4 \\ 1 & 0 & 3 \\ 5 & 4 & 6 \end{bmatrix}$ form a linearly independent set? Justify your answer.

Row reducing A to echelon form yields:

$$\begin{bmatrix} -4 & -3 & 0 \\ 0 & -1 & 4 \\ 1 & 0 & 3 \\ 5 & 4 & 6 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 3 \\ 0 & -1 & 4 \\ -4 & -3 & 0 \\ 5 & 4 & 6 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 3 \\ 0 & -1 & 4 \\ 0 & -3 & 12 \\ 0 & 4 & -9 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 3 \\ 0 & -1 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 7 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 3 \\ 0 & -1 & 4 \\ 0 & 0 & 7 \\ 0 & 0 & 0 \end{bmatrix}$$

Since there are no free variables, these vectors are linearly independent.