MATH 225 - May 28, 2008
Exam II - 50 minutes - 50 Points

NAME: Key

In order to receive full credit, SHOW ALL YOUR WORK. Full credit will be given only if all reasoning and work is provided. When applicable, please enclose your final answers in boxes.

1. (10 Points) Given the following third-order linear ordinary differential equation,

$$2\frac{d^3y}{dt^3} - 4\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 2y = 0. \tag{1}$$

and using the substitutions $\frac{d^2y}{dt^2} = a$ and $\frac{dy}{dt} = v$ rewrite the differential equation as a system of three first-order differential equations. Express this system in matrix-vector form.

$\frac{dy}{dt} = v \Rightarrow v' = a$

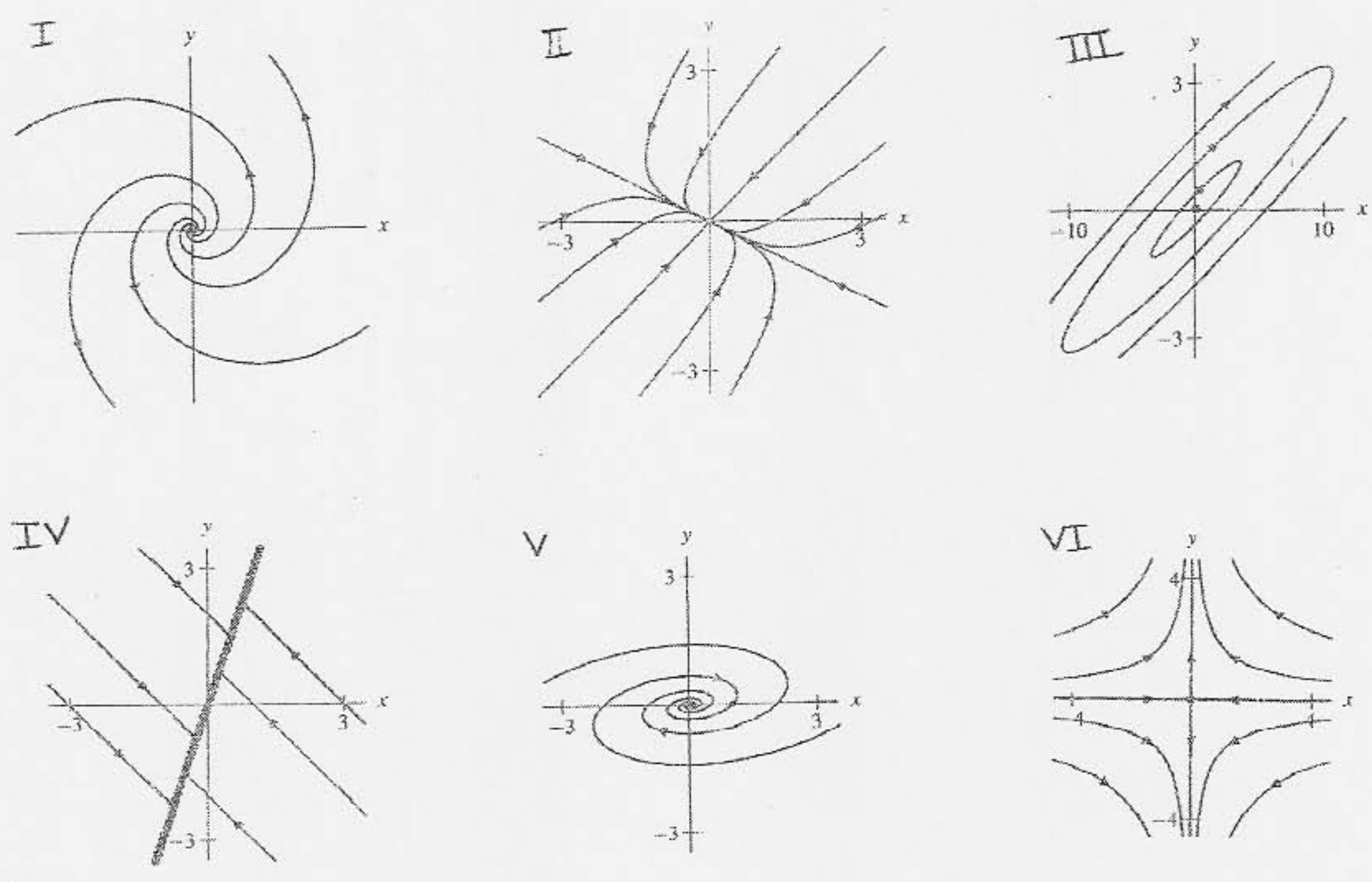
$\frac{d^2y}{dt^2} = a \Rightarrow a' = \frac{d^3y}{dt^3}$

$\Rightarrow (1) \Leftrightarrow a' = \frac{4}{2}a - 3v - 2y$ and $v' = a$, $y' = v$ $\Leftrightarrow \frac{d\vec{Y}}{dt} = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \vec{Y}$

2. (10 Points) Assume that the systems of differential equations (a)-(f) have the following eigenvalues,

(a) $\lambda_1 = -2i, \quad \lambda_2 = 2i,$ III	(b) $\lambda_1 = 3+i, \quad \lambda_2 = 3-i,$ I	(c) $\lambda_1 = \sqrt{3}, \quad \lambda_2 = -\sqrt{3},$ VI
(d) $\lambda_1 = 0, \quad \lambda_2 = -\pi,$ IV	(e) $\lambda_1 = -1-3i, \quad \lambda_2 = -1+3i,$ V	(f) $\lambda_1 = -1, \quad \lambda_2 = -2,$ II

Match the previous systems to their corresponding phase portraits,



3. (10 Points) Given the following linear system,

$$\frac{dx}{dt} = 3x - y \tag{2}$$

$$\frac{dy}{dt} = -x + 3y. \tag{3}$$

(a) Find all equilibrium points of the system.

$(x,y) = (0,0)$ is the only Equilibrium point.

(b) Calculate the eigenvalues of the system.

$$A = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} \Rightarrow \det(A - \lambda I) = (3-\lambda)^2 - 1 = \lambda^2 - 6\lambda + 9 - 1 = \lambda^2 - 6\lambda + 8 =$$

$$= (\lambda - 2)(\lambda - 4) = 0$$

$\lambda_1 = 2$

$\lambda_2 = 4$

(c) Calculate the eigenvectors of the system and find the general solution of the system.

Case $\lambda_1 = 2$:

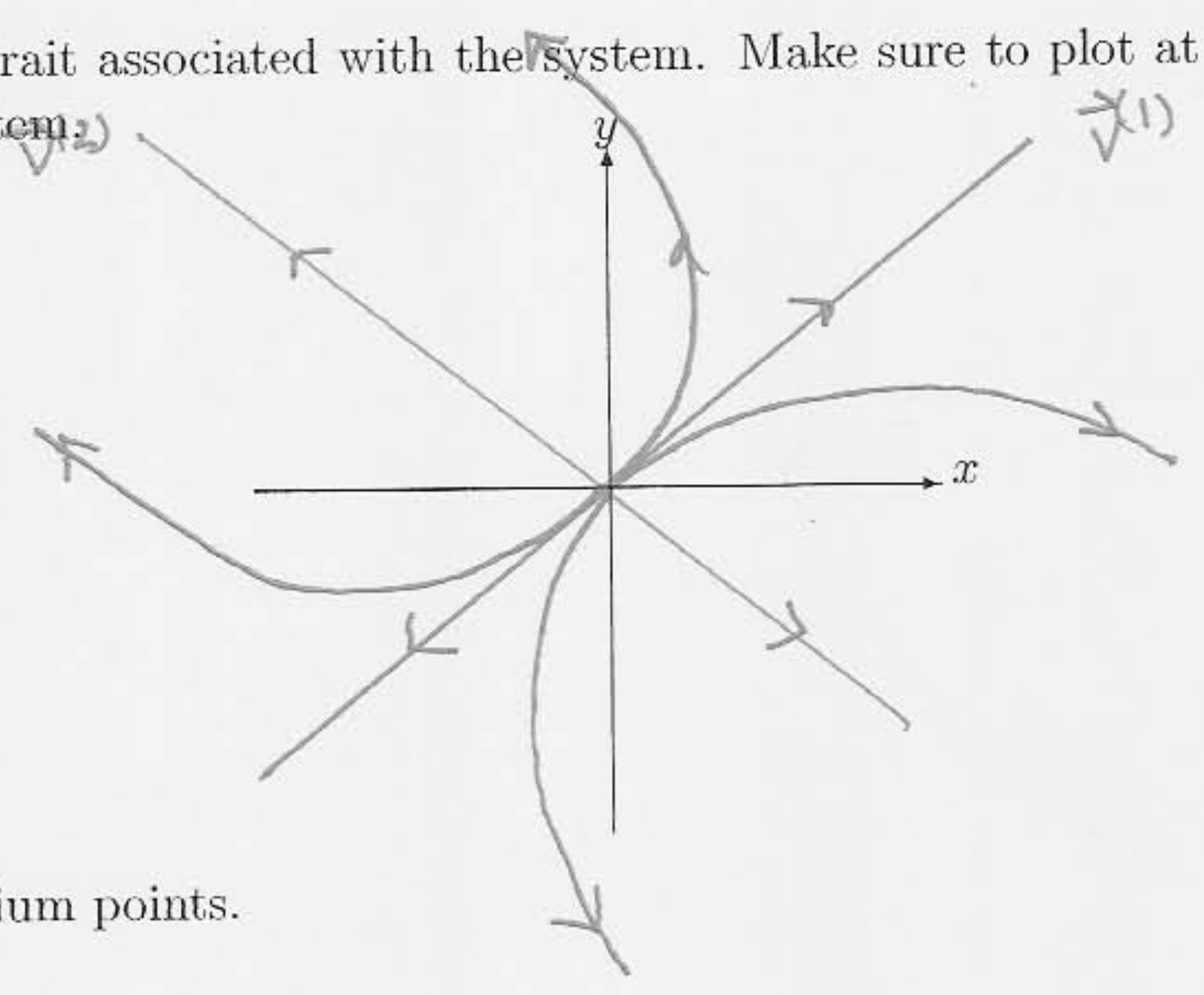
$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow v_1 = v_2 \Rightarrow \vec{V}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\vec{Y}(t) = k_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} + k_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{4t}$$

Case $\lambda_2 = 4$

$$\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow v_1 = -v_2 \Rightarrow \vec{V}^{(2)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

(d) Plot the phase portrait associated with the system. Make sure to plot at least four non-straight line solutions to the system.



(e) Classify all equilibrium points.

This is a Real unstable source

4. (10 Points) Solve the following system of differential equations and classify the equilibrium point.

$$\frac{d\mathbf{Y}}{dt} = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \mathbf{Y} \tag{4}$$

$$\det(A-\lambda I) = \det\left(\begin{bmatrix} -1-\lambda & 1 \\ -1 & -1-\lambda \end{bmatrix}\right) = (-1-\lambda)^2 + 1 = \lambda^2 - 2\lambda + 2 = 0$$

$$\lambda = \frac{+2 \pm \sqrt{4-4(2)}}{2} = 1 \pm i \Rightarrow \text{Spiral Source}$$

Case $\lambda_1 = 1+i$:

$$\begin{bmatrix} -1-(1+i) & 1 \\ -1 & -1-(1+i) \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow (-2+i)v_1 = -v_2 \Rightarrow \text{let } v_1 = 1 \Rightarrow v_2 = 2-i \Rightarrow \vec{V}^{(1)} = \begin{bmatrix} 1 \\ 2-i \end{bmatrix}$$

Thus,

$$\vec{Y}(t) = k_1 \begin{bmatrix} 1 \\ 2-i \end{bmatrix} e^{(1+i)t} + k_2 \begin{bmatrix} 1 \\ 2+i \end{bmatrix} e^{(1-i)t}$$

Or in Real Form:

$$\vec{Y}(t) = k_1 e^t \begin{bmatrix} \cos(t) \\ 2\cos(t)+\sin(t) \end{bmatrix} + k_2 e^t \begin{bmatrix} \sin(t) \\ 2\sin(t)-\cos(t) \end{bmatrix}$$

5. (10 Points) Solve the following initial value problem.

$$\frac{d\mathbf{Y}}{dt} = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} \mathbf{Y}, \quad \mathbf{Y}(0) = \begin{bmatrix} 5 \\ 2 \end{bmatrix} \tag{5}$$

$$\det(A-\lambda I) = +\lambda^2 + 4 = 0 \Rightarrow \lambda = \pm 2i$$

Case $\lambda_1 = 2i$

$$\begin{bmatrix} -2i & 1 \\ -4 & -2i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \vec{V}^{(1)} = \begin{bmatrix} 1 \\ 2i \end{bmatrix} \Rightarrow \vec{V}^{(2)} = \begin{bmatrix} 1 \\ -2i \end{bmatrix}$$

$\vec{V}^{(2)}$ not needed

$$\Rightarrow \vec{Y}(t) = k_1 \begin{bmatrix} \cos(2t) \\ -2\sin(2t) \end{bmatrix} + k_2 \begin{bmatrix} \sin(2t) \\ 2\cos(2t) \end{bmatrix}$$

$$\vec{Y}(0) = k_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} 0 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 5 \\ 2 \end{bmatrix} \Rightarrow \begin{matrix} k_2 = 1 \\ k_1 = 5 \end{matrix}$$

$$\vec{Y}(t) = \begin{bmatrix} 5\cos(2t) \\ -10\sin(2t) \end{bmatrix} + \begin{bmatrix} \sin(2t) \\ 2\cos(2t) \end{bmatrix}$$