(I) $u_t = c^2 u_{xx}$, $x \in \mathbb{R}$, $t \in \mathbb{R}^+$

Infinite domain

(II) $\lim_{x \to \pm\infty} u(x,t) = 0$

(III) $u(x,0) = f(x)$

Decay Condition w/ "nice" fn $\Rightarrow$ F.T. of u Exists

$$\mathcal{F}\{u_t - c^2 u_{xx}\} = \mathcal{F}\{u_t\} - c^2 \mathcal{F}\{u_{xx}\} =$$

$$= \frac{\partial}{\partial t}\mathcal{F}\{u\} + c^2 w^2 \mathcal{F}\{u\} =$$

$$= \frac{\partial \hat{u}}{\partial t} + c^2 w^2 \hat{u} = \mathcal{F}\{0\} = 0$$

You can only transform time or space.

Bounds of F.T. $\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\right]$ imply space.

$$\Rightarrow \underbrace{\frac{\partial \hat{u}}{\partial t}}_{\frac{d\hat{u}}{dt}} = -c^2 w^2 \hat{u}$$

where w is constant

$$\Rightarrow \hat{u}(w,t) = K(w)\, e^{-c^2 w^2 t}$$

PDE in Fourier domain.

<u>Key Point:</u> Only 1 type of derivative. $\Rightarrow$ treat as ODE.

<u>Note:</u> $\mathcal{F}\{\underbrace{u(x,0)}_{\hat{u}(w,0)}\} = \mathcal{F}\{f\} = \hat{f}(w) \Rightarrow \hat{u}(w,0) = \hat{f}(w) \Rightarrow K(w) = \hat{f}(w)$

<u>Soln in Fourier Space:</u> $\hat{u}(w,t) = \hat{f}(w)\, e^{-c^2 w^2 t} = \sqrt{2\pi}\, \hat{f}\, \frac{e^{-c^2 w^2 t}}{\sqrt{2\pi}}$

$$\mathcal{F}^{-1}\{\hat{u}(w,t)\} = u(x,t) = \mathcal{F}^{-1}\left\{\sqrt{2\pi}\, \hat{f}\, e^{-c^2 w^2 t}/\sqrt{2\pi}\right\} =$$

$$= \int_{-\infty}^{\infty} f(p) \frac{e^{-(x-p)^2/4c^2 t}}{2c\sqrt{\pi t}}\, dp$$

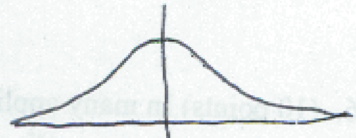
By convolution and last result where $a = c^2 t$

<u>Key Point:</u>

- Soln is initial state convolved/mixed with bell curves that both decay and spread out in time. [Notice as $t \to 0$ these bell curves become $\delta_0(x-p)$

$$\mathcal{F}\{e^{-ax^2}\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ax^2 - i\omega x}\,dx =$$

e^{-ax^2}



$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a\left(x^2 + \frac{i\omega}{a}x - \frac{\omega^2}{4a^2} + \frac{\omega^2}{4a^2}\right)}\,dx =$$

Complete the Square

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a\left[\left(x+\frac{i\omega}{2a}\right)^2 + \frac{\omega^2}{4a^2}\right]}\,dx$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{\omega^2}{4a}} \int_{-\infty}^{\infty} e^{-a(x+i\omega/2a)^2}\,dx$$

$u = x + \frac{i\omega}{2a}$

$du = dx$

Big lie! !! See MATH454 Complex Analysis. The bounds should be $\pm\infty + \frac{i\omega}{2a}$ Yikes.

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{\omega^2}{4a}} \int_{-\infty}^{\infty} e^{-au^2}\,du$$

$v = \sqrt{a}\,u$

$v^2 = a u^2$

$dv = \sqrt{a}\,du$

$$= \frac{1}{\sqrt{2\pi a}} e^{-\frac{\omega^2}{4a}} \underbrace{\int_{-\infty}^{\infty} e^{-v^2}\,dv}_{\sqrt{\pi}} = \frac{e^{-\frac{\omega^2}{4a}}}{\sqrt{2a}}$$

Integral from HW.

<u>Note:</u>

f	$\hat{f}$
e^{-ax^2}	$\dfrac{e^{-\omega^2/4a}}{\sqrt{2a}}$

for $a = 1/2$

$$\Rightarrow \mathcal{F}\{e^{-\frac{x^2}{2}}\} = e^{-\omega^2/2}$$

<u>Key Points:</u>

- Bell Curves transform to bell curves
- There is a bell curve that self transforms!