Nonlinear wave propagation.

- go from microscopic to macroscopic.

As before, we use Maxwell eqns to get wave eqn.

$$\underbrace{\nabla \times \nabla \times \vec{E}}_{\nabla(\nabla\cdot\vec{E}) - \nabla^2\vec{E}} + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{E} = -\frac{4\pi}{c}\frac{\partial^2 \vec{P}}{\partial t^2}$$

$$\nabla\cdot\vec{D} = \nabla\cdot(\epsilon\vec{E}) = 0$$

We'll assume $\epsilon \sim$ spatially constant, even though there is nonlinearity

Next separate $\vec{P} = \vec{P}^{(1)} + \vec{P}^{NL}$, bring linear part over.

$$\nabla^2\vec{E} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\underbrace{(\vec{E} + 4\pi\vec{P}^{(1)})}_{=\vec{D}^{(1)} = \epsilon^{(1)}\vec{E}} = \frac{4\pi}{c^2}\frac{\partial^2}{\partial t^2}\vec{P}^{NL}$$

assume isotropic

assume this is t-indep.

$$\nabla^2\vec{E} - \frac{\epsilon^{(1)}}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} = \frac{4\pi}{c^2}\frac{\partial^2\vec{P}^{NL}}{\partial t^2}$$

solutions:

linear eqn has wave solutions.

we will make the assumption that there will be several diff't waves at distinct frequencies.

e.g. $\vec{E}_n(\vec{r},t) = E_n(\vec{r})e^{-i\omega_n t} + c.c.$

$\vec{E}(\vec{r},t) = \sum_n \vec{E}_n(\vec{r},t) \qquad n \geq 0$

similar for $\vec{P}_n^{NL}$, and $\vec{D}_n^{(1)} = \epsilon_n^{(1)}\vec{E}_n^{(1)}$

$$\rightarrow -\nabla^2 \vec{E}_n(\vec{r}) - \frac{\omega_n^2}{c^2}\varepsilon^{(1)}(\omega_n)\vec{E}_n(\vec{r}) = \frac{4\pi\omega_n^2}{c^2}\vec{P}_n^{NL}(\vec{r})$$

What if material is birefringent?

$\vec{D}_n^{(1)} = \overleftrightarrow{\varepsilon}^{(1)}(\omega_n)\cdot\vec{E}_n^{(1)}$ since ε is a tensor

$\therefore$ change $\varepsilon \rightarrow \overleftrightarrow{\varepsilon}$ and dot it with $\vec{E}$

Notes: we have several coupled equations, each at diff't ω_n

- coupling is through $\vec{P}^{NL}$

e.g. $P^{(2)} = \chi^{(2)} E_m E_n^*$

each of these equations is for 3 vector components.

Application: SFG (sum freq. gen.)

- CW
- 2 inputs at ω_1, ω_2
- outputs at $\omega_3 = \omega_1 + \omega_2$
- plane waves prop in z direction

- ignore polarization effects for now

$E_n(z,t) = A_n e^{i(k_n z - \omega_n t)} + c.c.$

with no nonlinearity, no coupling of separate, linear w.e.

$\therefore$ all A_n's are constant

with nonlinearity, RHS $\rightarrow P^{NL}$ terms, e.g.

$$P_3^{(2)} = 4 d_{eff} E_1 E_2 = 4 d_{eff} A_1 A_2 e^{i(k_1+k_2)z}$$

recall $\chi^{(2)} \equiv 2 d_{eff}$

second 2x from $\omega_1+\omega_2$ or $\omega_2+\omega_1$

to put this into the NLWE, we have to account for $A_n(z)$, no dependence on x, y

$$-\frac{d^2}{dz^2}\vec{E}_3(z) - \frac{\omega_3^2}{c^2}\varepsilon_3\vec{E}_3(z) = \frac{16\pi\omega_3^2}{c^2} d_{eff}\vec{E}_1(z)\vec{E}_2(z)$$

Non-depleted pump approximation:

- input beams have powers $\propto A_1^2, A_2^2$
- anticipate growth of A_3 from zero, no initial change (much) of pump beams. A_1, A_2 const.

$\therefore$ equations are decoupled.

$$\rightarrow -\left(\frac{d^2}{dz^2}A_3\right)e^{ik_3z} - 2ik_3\left(\frac{d}{dz}A_3\right)e^{ik_3z} + \cancel{k_3^2 A_3 e^{ik_3z}} - \cancel{\frac{\varepsilon_3\omega_3^2}{c^2}A_3 e^{ik_3z}} = \frac{16\pi\omega_3^2}{c^2} d_{eff} A_1 A_2 e^{i(k_1+k_2)z}$$

cancellation from $k_3^2 = \varepsilon_3\omega_3^2/c^2$

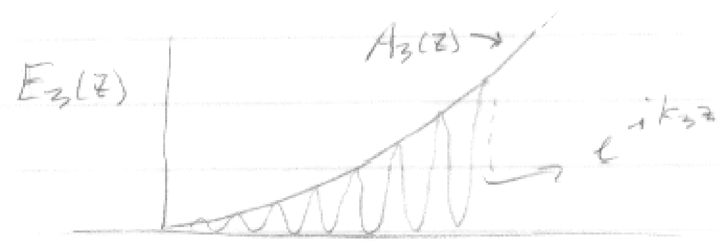
notice where e^{ik_3z} is: only on LHS

divide it out:

$$-\frac{d^2A_3}{dz^2} - 2ik_3\frac{dA_3}{dz} = \frac{16\pi\omega_3^2}{c^2} d_{eff} A_1 A_2 e^{i\Delta kz}$$

where $\Delta k \equiv k_1 + k_2 - k_3$ = phase mismatch.

Can we drop d^2A_3/dz^2 term?



let scale length for growth of A_3 be ℓ

$\rightarrow \frac{dA_3}{dz}$ scales as $\frac{1}{\ell}A_3$ e.g. $A_3 = A_{30}e^{z/\ell}$

compare terms:

$\frac{1}{\ell^2}A_3 : \frac{2\pi}{\lambda\ell}A_3$ if $\lambda\ell \ll \ell^2$ or $\lambda \ll \ell$

we can drop $d^2A_3/dz^2 \rightarrow$ slowly varying envelope approx

Effect of phase mismatch

- assume $\Delta k = 0 \rightarrow A_3(z) \propto z \quad I_3(z) \propto z^2$
- for $\Delta k \neq 0$:

$$\frac{dA_3}{dz} = s\, e^{i\Delta k z}$$

integrate directly:

$$A_3(z) = s \int_0^L e^{i\Delta k z}\, dz = s \left(\frac{e^{i\Delta k L} - 1}{i\Delta k}\right)$$

$$= s\, e^{i\Delta k L/2} \left(\frac{e^{i\Delta k L/2} - e^{-i\Delta k L/2}}{2i\Delta k L/2}\right)$$

$$= s\, e^{i\Delta k L/2} \frac{\sin(\Delta k L/2)}{\Delta k L/2} = \underline{s\, e^{i\Delta k L/2}\, \mathrm{sinc}\left(\frac{\Delta k L}{2}\right)}$$

Intensity is $\propto |A_3|^2$

$$I_i = \frac{n_i c}{8\pi} |\mathcal{E}|^2 \qquad E = \mathcal{E} \cos(kz - \omega t)$$

$$\qquad = A e^{i(kz - \omega t)} + c.c.$$

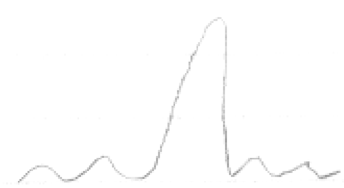
$$= \frac{n_i c}{2\pi} |A|^2 \qquad A = \mathcal{E}/2$$

$$\rightarrow I_3(L) = \frac{512\pi^5 d_{eff}^2 I_1 I_2}{n_1 n_2 n_3 \lambda_3^2 c} L^2\, \mathrm{sinc}^2(\Delta k L/2)$$

notes:
- $I_3 \propto d_{eff}^2$ magn. of deff is impt.
- I_3 is linear in I_1, I_2
- for $\Delta k \neq 0$, $I_3 \propto \frac{\sin^2(\Delta k L/2)}{(\Delta k)^2}$

power osc. along length.
- as $\Delta k \to 0$ period ↑
  ampl. ↑

- $\Delta k \neq 0$ fixed L vary Δk
  $I_3 \propto \mathrm{sinc}(\Delta k L/2)$



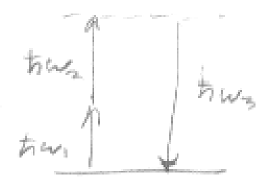
Conservation Laws

photon energy:

$$\left(-\nabla^2 + \frac{n_3^2}{c^2}\frac{\partial^2}{\partial t^2}\right) E_3 e^{-i\omega_3 t} = 4\pi \chi^{(2)} E_1 e^{-i\omega_1 t} E_2 e^{-i\omega_2 t}$$

to cancel time-dep. exponentials,

$$\hbar\omega_3 = \hbar\omega_1 + \hbar\omega_2$$



photon momentum

same argument if k's are all parallel

$$\hbar k_3 = \hbar k_1 + \hbar k_2$$

note that $\hbar k_i = \hbar n(\omega_i)\frac{\omega}{c}$

if $\omega_1 = \omega_2$,

$$\hbar k_3 = 2\hbar k_1$$

or $\frac{n(\omega_3)\omega_3}{c} = 2n(\omega_1)\frac{\omega_1}{c} \rightarrow \frac{1}{v_{ph}(\omega_3)} = \frac{1}{v_{ph}(\omega_1)}$

so phase vel. are equal if $\Delta k = 0$

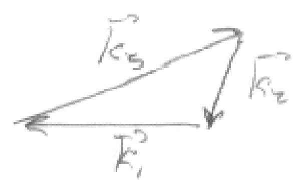
but for SFG this isn't right

$$\frac{\omega_3}{v_{ph3}} = \frac{\omega_1}{v_{ph1}} + \frac{\omega_2}{v_{ph2}}$$

so just remember $\Delta k = \sum k_i$

noncolinear case:

$$\vec{k}_3 = \vec{k}_1 + \vec{k}_2$$



beam power: Manley-Rowe

$$\frac{d}{dz}\left(\frac{I_1}{\omega_1}\right) = \frac{d}{dz}\left(\frac{I_2}{\omega_2}\right) = -\frac{d}{dz}\left(\frac{I_3}{\omega_3}\right)$$

intensity is a photon flux $\quad I_i = \frac{1}{2} v_i \cdot U_i = \frac{1}{2} \frac{c}{n_i} \frac{n_i^2 \mathcal{E}_i^{o2}}{8\pi}$

note location of index

$$I_i = \frac{\hbar\omega_i \cdot N_i}{\text{area} \cdot \text{time}} = \hbar\omega_i \cdot F_i$$

$\to$ photon flux.

$\therefore$ see that M-R relations result from photon conservation.

Solution of coupled mixing eqns – no phase mismatch

3-wave mixing, SVEA:

let $\xi = 8\pi d_{eff}/c$

$$A_1' = i\xi \frac{\omega_1}{n_1} A_3 A_2^* e^{-i\Delta k z}$$

$$A_2' = i\xi \frac{\omega_2}{n_2} A_3 A_1^* e^{-i\Delta k z}$$

$$A_3' = i\xi \frac{\omega_3}{n_3} A_1 A_2 e^{i\Delta k z}$$

valid for any input,

SFG: $\omega_1 + \omega_2 \to \omega_3$

DFG: $\omega_3 - \omega_2 \to \omega_1$

$\Delta k = k_1 + k_2 - k_3$

SHG: A_1, A_2 only $\quad 2\omega_1 \to \omega_2$

$$A_1' = i\xi \frac{\omega_1}{n_1} A_2 A_1^* e^{-i\Delta k z}$$

$\Delta k = 2k_1 - k_2$

$$A_2' = \frac{1}{2} i\xi \frac{\omega_3}{n_3} A_1^2 e^{i\Delta k z}$$

note factor of $\frac{1}{2}$

It is important to understand limiting cases, esp. $\Delta k = 0$

i) SFG upconversion: assume $A_2' = 0$ (small A_1, A_3; large A_2)

$$A_1' = i\xi \frac{\omega_1}{n_1} A_2^* A_3$$

$$A_1'' = i\xi \frac{\omega_1}{n_1} A_2^* \left(i\xi \frac{\omega_3}{n_3} A_2 A_1 \right) = \underbrace{-\xi^2 |A_2|^2 \frac{\omega_1 \omega_3}{n_1 n_3}}_{\equiv K^2 > 0,\ \underline{\text{real}}} A_1$$

$\to$ osc solutions

$$A_1(z) = B \cos Kz + C \sin Kz$$

$$A_1' = -KB \sin Kz + KC \cos Kz = i\xi \frac{\omega_1}{n_1} A_2^* A_3$$

$$A_3(z) = -\frac{iKn_1}{\xi \omega_1 A_2^*} \left(-B \sin Kz + C \cos Kz \right)$$

upconversion $A_3(0)=0 \rightarrow C=0 \qquad B=A_1(0)$

$A_1(z)=A_1(0)\cos Kz$

$$A_3(z)=\frac{iKn_1}{\xi\omega_1 A_2^*}A_1(0)\sin Kz = i\frac{\xi|A_2|\left(\frac{\omega_1\omega_3}{n_1n_3}\right)^{1/2}}{\xi(\omega_1/n_1)A_2^*}A_1(0)\sin Kz$$

$$=\underbrace{i\left(\frac{n_1\omega_3}{n_3\omega_1}\right)^{1/2}}_{\text{ampl. determines max conversion.}}A_1(0)\,e^{i\Phi_2}\sin Kz \qquad \text{let } A_2=|A_2|e^{i\Phi_2}$$

every photon at ω_3 uses a photon at ω_1

should have $n_1\frac{|A_1(z)|^2}{\omega_1}+n_3\frac{|A_3(z)|^2}{\omega_3}=n_1\frac{|A_1(0)|^2}{\omega_1}$

$$\checkmark \quad n_1\frac{|A_1(0)|^2}{\omega_1}\sin^2(Kz)+\frac{n_3}{\omega_3}\frac{n_1\omega_3}{n_3\omega_1}|A_1(0)|^2\cos^2 Kz=n_1\frac{|A_1(0)|^2}{\omega_3}$$

Note: > max conversion not set by d_{eff}

> but distance required to convert is $\propto 1/d_{eff}$

> If $A_1(0)=0$ no output. In principle, vacuum fluct. can be converted, but getting some signal at ω_3 doesn't increase rate.

> down conversion possible too.

2) DFG and parametric amp

here, assume strong pump at ω_3 $\quad A_3' = 0$

$$A_1' = i\xi \frac{\omega_1}{n_1} A_2^* A_3$$

$$A_2' = i\xi \frac{\omega_2}{n_2} A_1^* A_3 \rightarrow A_2'' = i\xi \frac{\omega_2}{n_2} A_3 \left(-i\xi \frac{\omega_1}{n_1} A_3^* A_2\right)$$

$$= +\underbrace{\xi |A_3|^2 \frac{\omega_1 \omega_2}{n_1 n_2}}_{K^2 > 0,\ \text{real}} A_2$$

now +'ve sign → exponential soln's.

- can seed with either one.
- exponential gain $\sinh(Kz)$ or $\cosh(Kz)$

  for every photon we make at ω_1, get one at ω_2

  - process builds on itself.

  ∴ high gain - can get optical parametric generation (OPG)

  - also spontaneous param. down conversion

  w/ no seed!

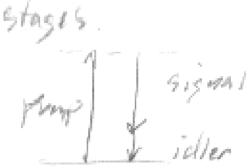
↑ ↓ $\quad \omega_1, \omega_2$ determined by $\Delta k = 0$

OPG can be a seed for an OPA (parametric amp)

- also use white-light from continuum.

can do multiple stages.

nomenclature



Spontaneous parametric down conversion: only pump, no seed (vacuum seeding).

- used for creating "entangled" photons. for quantum optics.
- properties of photons in pair are linked.

3) second harmonic generation

$$A_1' = i\xi \frac{\omega_1}{n_1} A_2 A_1^* e^{-i\Delta k z}$$

$$A_2' = \frac{1}{2} i \xi \frac{\omega_2}{n_2} A_1^2 e^{i\Delta k z}$$

this time do more to decouple eqns:

$$I = I_1 + I_2 \rightarrow \frac{c}{2\pi}\left(n_1 |A_1|^2 + n_2 |A_2|^2 \right) = I$$

make amplitudes dimensionless

$$A_i = \left(\frac{2\pi I}{n_i c}\right)^{\frac{1}{2}} a_i \qquad \text{so that } |a_1|^2 + |a_2|^2 = 1$$

$$\left(\frac{2\pi I}{n_2 c}\right)^{\frac{1}{2}} a_2' = i \frac{1}{2}\left(\frac{8\pi d}{c}\right)\left(\frac{2\pi I}{n_1 c}\right) \frac{\omega_2}{n_2} a_1^2 e^{i\Delta k z}$$

remaining constants must have dims of $1/\ell$

$$\text{let } \ell = \left(\frac{n_1^2 n_2 c^3}{2\pi I}\right)^{\frac{1}{2}} \frac{\frac{1}{2}\omega_2}{8\pi d_{eff}}$$

- Note that $\omega_1 = \frac{1}{2}\omega_2$ in this case, so both eqn.s have the same ℓ.
- scale z to ℓ: $\xi = z/\ell$ $\qquad \Delta s \equiv \Delta k \ell$

$$a_1' = i\, a_1^* a_2 e^{-i\Delta s \xi}$$

$$a_2' = i\, a_1^2 e^{i\Delta s \xi}$$

From here

1) numeric solution (see mixing solutions.nb)
2) analytic:
   let $a_i = u_i e^{i\phi_i}$ w/ u_i, ϕ_i real fcns.
   separate out Re, Im parts of eqns.