MATH332-Linear Algebra	**Homework Seven**

Least-Squares, Gram-Schmidt, Orthogonal Diagonalization, SVD

Text: Chapter 6-7 Section Overviews: 6.1-6.6, 7.1-7.2

Quote of Homework Seven	
Take a wrinkled raisin and do with it what you will.	
	Ween : Roses are Free (1994)

1. Geometry in $\mathbb{R}^n$

1.1. **Parallelogram Identity.** Let $\mathbf{u}$,$\mathbf{v}\in\mathbb{R}^n$. Prove that $||\mathbf{u}+\mathbf{v}||^2+||\mathbf{u}-\mathbf{v}||^2=2||\mathbf{u}||^2+2||\mathbf{v}||^2$.

1.2. **Orthogonal Complements.** Let W be a subspace of $\mathbb{R}^n$. Prove that $\mathrm{W}^\perp$ is a subspace of $\mathbb{R}^n$.

1.3. **Length Invariance.** Let $\mathbf{U}$ be an orthogonal matrix. Prove that $||\mathbf{Ux}||=||\mathbf{x}||$.

1.4. **Angle Invariance.** Let $\mathbf{U}_{n\times n}$ be an orthogonal matrix and $\mathbf{x},\mathbf{y}\in\mathbb{R}^n$. Prove that $\mathbf{Ux}\cdot\mathbf{Uy}=\mathbf{x}\cdot\mathbf{y}$.

1.5. **Orthogonality Invariance.** Let $\mathbf{U}_{n\times n}$ be an orthogonal matrix and $\mathbf{x},\mathbf{y}\in\mathbb{R}^n$. Prove that $\mathbf{Ux}\cdot\mathbf{Uy}=0$ if and only if $\mathbf{x}\cdot\mathbf{y}=0$.

2. Orthogonal Projections

Given,

$$\mathbf{y}=\begin{bmatrix}4\\8\\1\end{bmatrix},\quad \mathbf{u}_1=\begin{bmatrix}\frac{2}{3}\\\frac{1}{3}\\\frac{2}{3}\end{bmatrix},\quad \mathbf{u}_2=\begin{bmatrix}-\frac{2}{3}\\\frac{2}{3}\\\frac{1}{3}\end{bmatrix}.$$

2.1. **Products of Rectangular Orthogonal Matrices.** Let $\mathbf{U}=[\mathbf{u}_1\quad\mathbf{u}_2]$. Compute $\mathbf{U}^{\mathrm{T}}\mathbf{U}$ and $\mathbf{U}\mathbf{U}^{\mathrm{T}}$.

2.2. **Projections.** Let $\mathrm{W}=\mathrm{span}\{\mathbf{u}_1,\mathbf{u}_2\}$. Compute $\mathrm{proj}_{\mathrm{W}}\,\mathbf{y}$ and $(\mathbf{U}\mathbf{U}^{\mathrm{T}})\mathbf{y}$.

2.3. **Vector Decomposition.** Write $\mathbf{y}$ as the sum of a vector $\hat{\mathbf{y}}$ in W and a vector $\mathbf{z}$ in $\mathrm{W}^\perp$.

2.4. **Geometric Consequences.** Describe the geometric relationship between the plane W in $\mathbb{R}^3$ and the vectors $\hat{\mathbf{y}}$ and $\mathbf{z}$ from part 2.3.

3. QR Factorization

Given,

$$\mathbf{A}_1=\begin{bmatrix}1&2&5\\-1&1&-4\\-1&4&-3\\1&-4&7\\1&2&1\end{bmatrix},\quad \mathbf{A}_2=\begin{bmatrix}2&3\\2&4\\1&1\end{bmatrix},\quad \mathbf{b}=\begin{bmatrix}7\\3\\1\end{bmatrix}.$$

3.1. **QR Me One.** Determine the $\mathbf{QR}$ factorization of $\mathbf{A}_1$.

3.2. **Linear Independence.** Show that the columns of $\mathbf{A}_2$ are linearly independent.

3.3. **QR Me Two.** Determine the $\mathbf{QR}$ factorization of $\mathbf{A}_2$.

3.4. **Least-squares.** Using this factorization calculate the unique least-squares solution $\hat{\mathbf{x}}=\mathbf{R}^{-1}\mathbf{Q}^{\mathrm{T}}\mathbf{b}$.[1]

[1] See theorem 6.5.15 on page 414.

4. Least-Squares

Recall the interpolation problem from an earlier homework:
Suppose you have a set S of three points in $\mathbb{R}^2$,

$$S_1 = \{(1,12),(2,15),(3,16)\}$$
$$S_2 = \{(1,12),(1,15),(3,16)\}$$
$$S_3 = \{(1,12),(2,15),(2,15)\}$$

which you seek to interpolate with the quadratic polynomial $p(t) = a_0 + a_1 t + a_2 t^2$.

4.1. **Least-Squares Approximation.** Find the least-squares solution for S_3 which we previously found to have no solution.

4.2. **Least-Squares Geometry.** Given the linear system of equations,

$$x_1 + x_2 = 2$$
$$x_1 + x_2 = 4$$

4.2.1. *Least-Squares.* Determine the least-squares solution to the linear system.

4.2.2. *Least-Squares Error.* Determine the least-squares error associated with the linear system.

4.2.3. *Graphical Interpretation.* Graph the linear system, the least-squares solution, and the least-squares error in $\mathbb{R}^2$.

5. Singular Value Decomposition

Given,

$$\mathbf{A} = \begin{bmatrix} 7 & 1 \\ 0 & 0 \\ 5 & 5 \end{bmatrix}.$$

5.1. **Do it!** Find a singular value decomposition of $\mathbf{A}$.

6. EXTRA CREDIT - Gram-Schmidt in Function Spaces

In homework 6 we showed that the first four Hermite polynomials were linearly independent and thus a basis for $\mathbb{P}_3$.[2] While this makes good use of the material from 4.4 outside of the context of $\mathbb{R}^n$ it really misses the point.[3] The Hermite polynomials are orthogonal polynomials and constitute an orthonormal basis for vector space $L^2(-\infty,\infty)$.[4] To see why this is true we must define the inner-product to be,

$$f \cdot g = \int_{-\infty}^{\infty} f(x)g(x)e^{-x^2}\,dx, \tag{1}$$

which is different than our standard definition in $\mathbb{R}^n$. [8] We take without proof that this definition satisfies the axioms of an inner-product. Recall the first few Hermite Polynomials, [9]

$$H_0(x) = 1, \quad H_1(x) = 2x, \quad H_2(x) = -2 + 4x^2, \quad H_3(x) = -12x + 8x^3, \quad x \in (-\infty,\infty),$$

satisfying the Rodrigues representation,

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}. \tag{2}$$

6.1. **Symmetric of the Hermite Polynomials.** Prove that $H_{2n}(x)$ is and even function and that $H_{2n+1}(x)$ is an odd function. [10]

6.2. **An Orthogonality Result.** Prove that the even Hermite polynomials are orthogonal to the odd Hermite polynomials.

6.3. **Gram-Schmidt Part I.** Normalize H_0 and H_1.

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[2]We take without proof that the first $n+1$ Hermite polynomials are linearly independent and thus a basis for $\mathbb{P}_n$.

[3]The Hermite polynomials are prevalent in statistics, applied mathematics and physics but not in the context of polynomial spaces.

[4]The vector space $L^2(-\infty,\infty)$ is an infinite dimensional complete inner-product space or a Hilbert space, in honor of David Hilbert `http://en.wikipedia.org/wiki/David_Hilbert`. The space L^2, which is an abstraction of standard Euclidean space, is important because its elements must have finite length and any infinite-sequence of elements must converge to a point in L^2. The condition that 'vectors' must have finite length typically implies that they have finite energy, which is what one would hope. While, the convergence properties allows use to take limits without leaving the space.[6]

6.4. **Gram-Schmidt Part II.** Using the normalized Hermite polynomials apply Gram-Schmidt and find $H_2(x)$. [11]

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[6]Indeed, things would be very bad if this were not the case. Consider the infinite sum, $\sum_{n=0}^{\infty} \frac{4(-1)^n}{2n+1}$. The summands are all rational but this sum converges to π, which is irrational. That is, the rationals are not closed under limits of arbitrary linear combinations![7]

[7]Yeah, I footnoted a footnote. What of it?!

[8]If we used the standard inner-product and made the Hermite polynomials an orthonormal basis, via GramSchmidt, for $\mathbb{P}_n$ then we would have gotten to the standard polynomial basis, which is nothing new.

[9]For more we can look at `http://en.wikipedia.org/wiki/Hermite_polynomials`. There are, in general, infinitely-many of them arising as eigenfunctions of the differential operator $\dfrac{d^2}{dx^2} - x\dfrac{d}{dx}$.

[10]Recall that an even function has the property that $f(-x) = f(x)$ and an odd function has the property that $f(-x) = -f(x)$. To make this clear from Rodrigues representation you should show that the derivative of an even function is an odd function and that the derivative of an odd function is an even function.

[11]MIT's open courseware site has a nice discussion of GS applied to the Legendre polynomials. `web.mit.edu/18.06/www/Spring09/legendre.pdf` To do this first consider a general quadratic, $H_2(x) = ax^2 + bx + c$, and argue that $b = 0$. Next, we want to find a and c such that $H_2(x)$ is orthogonal to $H_1(x)$ and $H_0(x)$. Gram-Schmidt gives us a formula for this, page 404 of the text, only every inner-product must be thought of in the sense of (1). After this calculation you should have a relation between a and c. To find 'a' normalize $H_2(x)$ and compare your result to $H_2(x)$ as it is given. They should look the same up a multiplicative constant.