MATH 348 - Advanced Engineering Mathematics November 14, 2007
Homework 8, Fall 2007 **Due**: December 5, 2007

LINEAR ALGEBRA - VECTOR SPACES, DETERMINANTS, INVERSE MATRICES AND EIGENVALUE PROBLEMS

1. Given,

$$\mathbf{A} = \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ -2 & 3 & -3 & -3 & -4 \\ 4 & -6 & 9 & 5 & 9 \\ -2 & 3 & 3 & -4 & 1 \end{bmatrix}.$$

   (a) What is the basis and dimension of Nul $\mathbf{A}$?
   (b) What is the basis and dimension of Col $\mathbf{A}$?
   (c) What is the basis and dimension of Row $\mathbf{A}$?
   (d) What is the Rank of $\mathbf{A}$?

2. Given,

$$\mathbf{A} = \begin{bmatrix} 3 & 6 & 7 \\ 0 & 2 & 1 \\ 2 & 3 & 4 \end{bmatrix}.$$

   Determine $\mathbf{A}^{-1}$ via:

   (a) Calculate $\det(\mathbf{A})$.
   (b) The Gauss-Jordan Method (pg.317).
   (c) The cofactor representation (Theorem 2 pg.318).
   (d) Check your result by showing $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$

3. Given,

$$\mathbf{A} = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}.$$

   (a) Determine the eigenvalues of $\mathbf{A}$.
   (b) Determine the eigenvectors of $\mathbf{A}$.

   Hint: Use a cofactor expansion along the first row to get $(4-\lambda)(1-\lambda)^2 + 2(1-\lambda) = 0$.

4. Given,

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}. \tag{1}$$

   Calculate $\mathbf{A}^{1000}$ and $\mathbf{A}^{1001}$.

5. Let,

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{2}$$

   Show that the eigenvectors of $\mathbf{A}$ are orthonormal.

3. $A = \begin{bmatrix} 2 & -3 & 6 & 2 & +5 \\ -2 & 3 & -3 & -3 & -4 \\ 4 & -6 & 9 & 5 & 9 \\ -2 & 3 & 3 & -4 & 1 \end{bmatrix} \underset{\substack{R3 = R3 - 2R1 \\ R4 = R4 + R1}}{\overset{R2 = R2 + R1}{\sim}} \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ 0 & 0 & 3 & -1 & 1 \\ 0 & 0 & -3 & 1 & -1 \\ 0 & 0 & 9 & -2 & 6 \end{bmatrix} \sim$

$\sim \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ 0 & 0 & +3 & -1 & +1 \\ 0 & 0 & -3 & +1 & -1 \\ 0 & 0 & 9 & -2 & 6 \end{bmatrix} \underset{R4 = R4 + 3R3}{\overset{R3 = R3 + R2}{\sim}} \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ 0 & 0 & +3 & -1 & +1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix} \sim$

$$\sim \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ 0 & 0 & 3 & -1 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ \underbrace{0}_{P.C} & 0 & \underbrace{0}_{P.C} & \underbrace{0}_{P.C} & 0 \end{bmatrix} = B$$

a. $B \Rightarrow A\vec{x} = 0$ has the general solution set,

$x_4 = -3x_5$

$x_3 = \frac{(x_4 - x_5)}{3} = \frac{(-3x_5 - x_5)}{3} = \left(\frac{-4}{3}\right)x_5$

$x_1 = \frac{1}{2}(3x_2 - 6x_3 - 2x_4 - 5x_5) = \frac{1}{2}(3x_2 - 6(-\frac{4}{3}x_5) - 2(-3x_5) - 5x_5) =$

$= \frac{1}{2}(3x_2 + 8x_5 + 6x_5 - 5x_5) = \frac{3}{2}x_2 + \frac{9}{2}x_5$

$x_2 = \text{free}$

$x_5 = \text{free}.$ $\Rightarrow \vec{x} = x_2 \begin{bmatrix} 3/2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 9/2 \\ 0 \\ -4/3 \\ -3 \\ 1 \end{bmatrix}$, $x_2, x_5 \in \mathbb{R}$

Thus the Basis for Nul A is

$$B_{Null} = \left\{ \begin{bmatrix} 3/2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 9/2 \\ 0 \\ -4/3 \\ -3 \\ 1 \end{bmatrix} \right\}$$

and $\dim(\text{Nul}A) = \dim B_{Null} = 2$

b. B ⇒ That the basis for the column space of A is the pivot columns $\vec{a}_1, \vec{a}_3, \vec{a}_4$ of A.

$$B_{ColA} = \left\{ \begin{bmatrix} 2 \\ -2 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} 6 \\ -3 \\ 9 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 5 \\ -4 \end{bmatrix} \right\}$$

and

$\dim(\text{Col}A) = \dim B_{ColA} = 3$

c. B ⇒ The Basis for Row A is given as

$$B_{RowA} = \left\{ \begin{matrix} [2, -3, 6, 2, 5], \\ [0, 0, 3, -1, 1], \\ [0, 0, 0, 1, 3] \end{matrix} \right\}$$

$\dim(\text{Row}A) = \dim B_{RowA} = 3.$