12 / 4 / 06

Note Title 11/20/2006

Solutions of Laplace Eqn

$$\nabla^2 \psi(r,\theta,\varphi) = 0$$

$$\psi(r,\theta,\varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left(A_{lm} r^{l} + B_{lm} r^{-(l+1)}\right) Y_{lm}(\theta,\varphi)$$

$$Y_{lm}(\theta,\varphi) = N\, P_{lm}(\theta)\, e^{im\varphi}$$

Normalization const

$$N = \sqrt{\frac{(2l+1)}{4\pi} \frac{(l-m)!}{(l+m)!}}$$

$$Y_{00}(\theta,\varphi) = \frac{1}{\sqrt{4\pi}}$$

$$Y_{10}(\theta,\varphi) = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_{1\,\pm1}(\theta,\varphi) = \mp\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\pm\varphi}$$

$$Y_{20}(\theta,\varphi) = \sqrt{\frac{5}{16\pi}}\left(2\cos^2\theta - \sin^2\theta\right)$$

$$Y_{2\,\pm1}(\theta,\varphi) = \mp\sqrt{\frac{15}{8\pi}}\cos\theta\sin\theta\, e^{\pm i\varphi}$$

$$Y_{2\,\pm2}(\theta,\varphi) = \sqrt{\frac{15}{32\pi}}\sin^2\theta\, e^{\pm 2i\varphi}$$

$$\int_0^{2\pi}\int_0^{\pi} Y_{\ell m}(\theta,\varphi)\, Y^{*}_{\ell' m'}(\theta,\varphi)\, \sin\theta \, d\theta \, d\varphi$$

$$= \delta_{\ell\ell'}\,\delta_{mm'}$$

$$\sin\theta \, d\theta \, d\varphi \equiv d\Omega$$ solid angle

$$= \int_{4\pi} Y_{\ell m}(\theta,\varphi)\, Y^{*}_{\ell' m'}(\theta,\varphi)\, d\Omega$$

$$= \delta_{\ell\ell'}\,\delta_{mm'}$$

The spherical harmonics are the natural functions to use on spheres

1) They are orthogonal

2) They are complete

i.e. any function that is well-behaved on the sphere, $f(\theta,\varphi)$ can be written

$$f(\theta,\varphi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} A_{\ell m} Y_{\ell m}(\theta,\phi)$$

As usual to get the expansion coeff. we project

$$\boxed{A_{\ell m} = \int_{4\pi} f(\theta,\varphi)\, Y^*_{\ell m}(\theta,\varphi)\, d\Omega}$$

E.g. suppose $f(\theta,\varphi) = 1$

$$A_{\ell m} = \int_{4\pi} 1 \; Y^*_{\ell m}(\theta,\varphi)\, d\Omega$$

well $1 = \dfrac{\sqrt{4\pi}}{\sqrt{4\pi}} = \sqrt{4\pi}\, Y_{00}$

$$A_{\ell m} = \sqrt{4\pi} \int_{4\pi} Y_{00}\, Y^*_{\ell m}\, d\Omega$$

$$= \sqrt{4\pi}\, \delta_{\ell 0}\, \delta_{m0}$$

$$\Rightarrow \quad A_{\ell m} = \sqrt{4\pi}\, \delta_{\ell 0}\, \delta_{m 0}$$

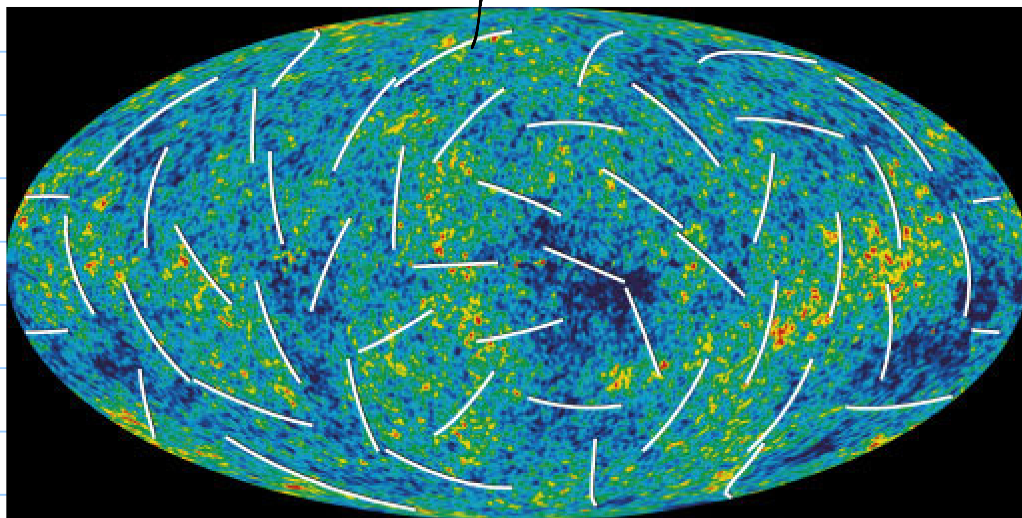
or $$A_{00} = \sqrt{4\pi}$$

all other $A_{\ell m} = 0$

so $$1 = \sqrt{4\pi}\, Y_{00}$$

24- 100 GHz

polarization direct.

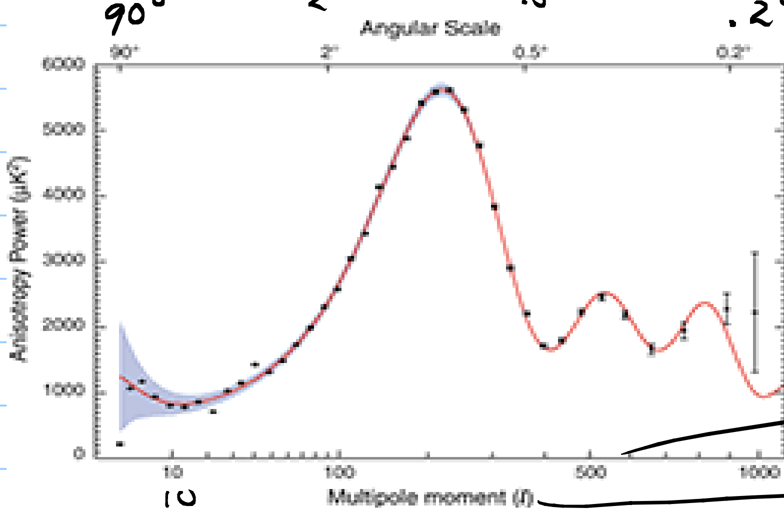


$$I(\theta,\phi) = \sum_{\ell=0}^{1000} \sum_{m=-\ell}^{\ell} A_{\ell m} Y_{\ell m}(\theta,\phi)$$

wilkinson microwave
anisotropy probe

shorter wavelengths red
longer blue

90° 2° .5° .2°

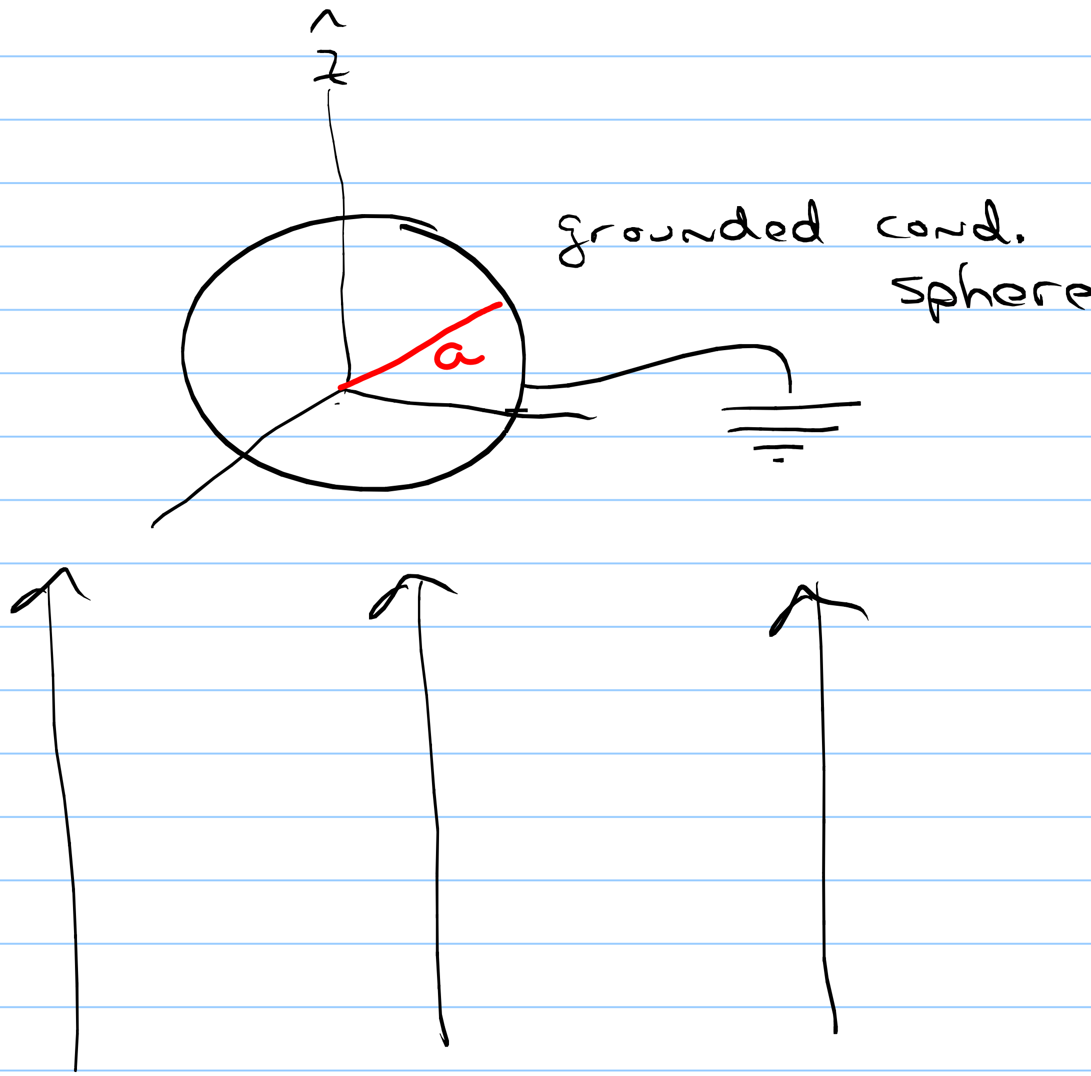


10

500

ℓ

Axisymmetric ex.



$$\vec{E} = E_0 \hat{z}$$

Axial symmetry $\Rightarrow m=0$

So our solution to $\nabla^2 V = 0$

remember $E = -\nabla V$

so $V = -E_0 z$

$$V(r,\theta,\phi) = \sum \left(A_{\ell m} r^{\ell} + B_{\ell m} r^{-(\ell+1)}\right) Y_{\ell m}(\theta,\phi)$$

$$\Downarrow$$

$$V(r,\theta) = \sum_{\ell=0}^{\infty} \left(A_\ell r^{\ell} + B_\ell r^{-(\ell+1)}\right) P_\ell(\cos\theta)$$

Since sphere is conducting

BC 1 $V(r=a,\theta) = 0$

$$= \sum_{\ell=0}^{\infty} \left(A_\ell a^{\ell} + B_\ell a^{-(\ell+1)}\right) P_\ell = 0$$

BC 2 Since $\bar{E}$ as $r \to \infty = E_0 \hat{z}$

$$\Rightarrow V(r\to\infty,\theta) = -E_0 z$$

BC 1

multiply both sides by $P_{\ell'}(\cos\theta)$ integrate

$$\sum_{\ell=0}^{\infty} \left(A_\ell a^\ell + B_\ell a^{-(\ell+1)}\right) \underbrace{\int_{-1}^{1} P_\ell P_{\ell'} \, dx}_{\frac{2}{2\ell+1}\delta_{\ell\ell'}}$$

$$\Rightarrow \quad A_\ell a^\ell + B_\ell a^{-(\ell+1)} = 0$$

$$\Rightarrow \quad B_\ell = -\frac{a^\ell}{a^{-(\ell+1)}} A_\ell = -a^{2\ell+1} A_\ell$$

Now B.C. 2 which applies as $r \to \infty$

$$V(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + B_l r^{-(l+1)}\right) P_l(\cos\theta)$$

note that for large r
only the A_l terms survive
so a $r \to \infty$ B.C. does
not constrain B_l

so

$$\lim_{r \to \infty} V(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

$$= -E_0 z$$

$$= -E_0 r \cos\theta$$

$P_1(x) = x$ or $P_1(\cos\theta) = \cos\theta$

so $-E_0 z = -E_0 r \cos\theta$

so only the $l=1$ term is involved

$$\lim_{r \to \infty} V(r,\theta) = A_1 r P_1(\cos\theta)$$
$$= -E_0 r P_1(\cos\theta)$$

$$\Rightarrow \boxed{A_1 = -E_0}$$

BC 1 said

$$B_l = -a^{2l+1} A_l$$
$$\Rightarrow B_1 = -a^3(-E_0)$$

Putting this all together

$$V(r,\theta) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + B_\ell r^{-(\ell+1)}\right) P_\ell$$

$$= \left(A_1 r + B_1 r^{-2}\right) P_1$$

$$= \left(-E_0 r + a^3 E_0 r^{-2}\right) \cos\theta$$

$$= -E_0 \left(1 - \frac{a^3}{r^3}\right) r \cos\theta$$

Exercise, compute ∇ is spherical coord

You'll get

$$E_r = E_0 \left(1 + 2\left(\frac{a}{r}\right)^3\right) \cos\theta$$

$$E_\theta = -E_0 \left(1 - \left(\frac{a}{r}\right)^3\right) \sin\theta$$

check out

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