Linear systems

$$\xrightarrow{f(t)} \boxed{S} \xrightarrow{g(t)}$$

$f(t) \to F(\omega)$

$g(t) \to G(\omega)$

how are all these related?

$g(t) = \hat{S} f(t)$ $\hat{S}$ = operator for the system

linear if $\hat{S}\left(a_1 f_1(t) + a_2 f_2(t)\right) = a_1 \hat{S}(f_1) + a_2 \hat{S}(f_2)$

i.e. superposition holds

shift invariant if $\hat{S}\left(f(t-t_0)\right) = g(t-t_0)$

in the time domain $\hat{S}$ is time independent. "time-invariant"

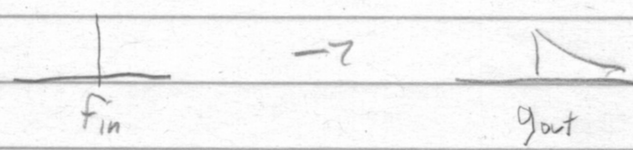
spatial " $\hat{S}$ is the same for all $\vec{r}$

LSI system is both: linear, shift invariant

= LTI system in t-ω space.

Causality:

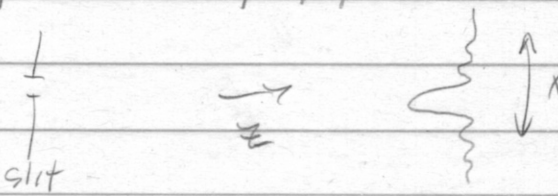
- in time domain what happens at time t not influenced by later times $> t$



- spatial domain - different

e.g. forward propagation z is like t

no back reflections



> system past can influence present. resonances, delayed respon

General input (any $f(t)$)

can always write this

$$g(t) = \hat{\mathcal{L}} f(t) = \hat{\mathcal{L}} \int f(t')\,\delta(t-t')\,dt'$$

$\hat{\mathcal{L}}$ operates on functions of t, not t'

move $\hat{\mathcal{L}}$ inside

$$\therefore \quad g(t) = \int f(t') \left(\hat{\mathcal{L}}\, \delta(t-t') \right) dt'$$

$$= \int f(t')\, h(t-t')\, dt'$$

$$= f \otimes h$$

☆ convolve input w/ impulse response to get output.

in frequency space

$$G(\omega) = \mathcal{F}\{f \otimes h\} = \underline{H(\omega)}\, F(\omega)$$

↳ transfer function.

<u>implications</u>

system characterization:

1) input ω → measure $A(\omega)$, $\phi(\omega)$
→ $H(\omega)$ → predict output $g(t)$ for any $f(t)$

2) input $\delta(t)$ measure output $h(t)$ impulse response
→ $H(\omega) = \mathcal{F}\{h(\omega)\}$

applications: filters, signal processing, image processing,
phase effects, control in ultrafast pulses

Impulse response

δ-function input $\quad \delta(t)$ = ultrashort pulse

$\delta(x,y)$ = point source.

$$h(t) = \hat{\mathcal{L}}\, \delta(t) \quad = \text{impulse response}$$

$$h(t-t_0) = \hat{\mathcal{L}}\, \delta(t-t_0) \quad \text{same fcn } h(\,) \text{ if shift invariant}$$

impulse response contains all info on LSI systems.

Transfer functions

suppose $f(t) = e^{-i\omega_0 t}$ input a pure frequency

$$g(t) = \hat{\mathcal{L}}\, f(t) = A e^{i\phi} e^{-i\omega_0 t}$$
$$= H(\omega_0)\, f(t) \qquad H(\omega_0) = A e^{i\phi} \text{ a complex number}$$

in words $e^{-i\omega_0 t}$ = eigenfunction

$H(\omega_0)$ = eigenvalue (complex)

$A(\omega_0)$ = amplitude factor (gain or loss)

$\phi(\omega_0)$ = phase shift

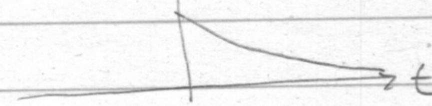
A, ϕ are real functions

output is the same ω_0, but ampl. and phase are changed.

$H(\omega) \equiv$ transfer function

Examples:

$$h(t) = e^{-\gamma t}\,\Theta(t)$$



$$H(\omega) = \mathcal{F}\{h(t)\}$$

$$= \int_0^{\infty} e^{-\gamma t} e^{i\omega t}\,dt = \frac{1}{i\omega-\gamma}(0-1)$$

$$= \frac{1}{\gamma - i\omega} = \frac{\gamma}{\gamma^2+\omega^2} + i\frac{\omega}{\gamma^2+\omega^2}$$

complex Lorentzian.

If $h(t)$ is real then $H(\omega) = H^*(-\omega)$ hermitian.

$$H(\omega) = \int_{-\infty}^{\infty} h(t)\,e^{i\omega t}\,dt$$

$$H^*(-\omega) = \int h(t)\,e^{-i(-\omega)t}\,dt = H(t)$$

For complex Lorentzian above, this checks out.

Note $Re(H(\omega))$ is even in ω

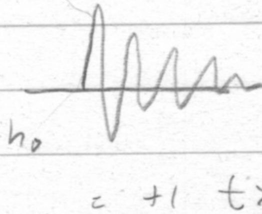
$Im(H(\omega))$ is odd

This observation is related to the Kramers-Kronig relations.

when a system is <u>causal</u> $h(t) = 0$ for $t<0$

$h(t) =$ real for $t \geq 0$

e.g.



we can write $h(t) = h_e + h_o$

and $h_e = \underbrace{\text{signum}(h_o)}$ $= +1 \quad t>0$

$-1 \quad t<0$

this relation between the even and odd parts of $h(t)$ leads to a relation between the real and imaginary parts of $H(\omega)$

Physically: measuring absorption spectrum
$\rightarrow$ characterizing refractive index.

example 2

$$h(t) = \Theta(t)\, e^{-\gamma t} \sin \omega_0 t$$

$$H(\omega) = \frac{1}{\gamma - i\omega} \otimes \underbrace{\mathcal{F}\{\sin \omega_0 t\}}_{\frac{1}{2i}\left(\delta(\omega+\omega_0) - \delta(\omega-\omega_0)\right)}$$

$$= \frac{1}{2i}\left[\frac{1}{\gamma - i(\omega+\omega_0)} - \frac{1}{\gamma - i(\omega-\omega_0)}\right]$$

Characterization of linear systems.

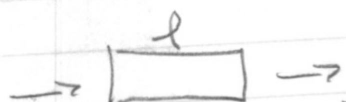
1) directly measure transfer fcn $H(\omega)$
 source - tune ω_{in}
 measure amplitude $A(\omega)$
 measure phase shift $\Phi(\omega)$ (need a way)

 or put in all ω's at once (white light, noise)

2) measure impulse response $h(t)$
 short pulse input
 measure $a(t)$, $\Phi(t)$

FT or FT^{-1} to get results in other domain.

Physical example:
 complex refractive index
 $$n(\omega) = n_R(\omega) + i\, n_I(\omega)$$



$e^{-i\omega t}$ $\qquad$ $e^{-i\omega t}\, e^{ikl}$ $\qquad$ $k = \dfrac{\omega\, n(\omega)}{c}$

$$e^{-i\omega t}\, \underbrace{e^{i \frac{\omega}{c} n_R(\omega) l}\; e^{-\frac{\omega}{c} n_I(\omega) l}}_{H(\omega)}$$

$$\left.\begin{aligned} \text{phase shift, } \quad \Phi(\omega) &= \frac{\omega}{c}\, n_R(\omega)\, l \\ A(\omega) &= e^{-\frac{\omega}{c} n_I(\omega) l} \end{aligned}\right\} H(\omega) = A e^{i\Phi}$$

absorption if $n_I(\omega) > 0$
gain if < 0