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1. Let Λ be a nonempty indexing set and let $\mathcal{A} = \{A_\alpha \,|\, \alpha \in \Lambda\}$ be an indexed family of sets. Also, assume that $\Gamma \subseteq \Lambda$ and that $\Gamma \neq \varnothing$.
   Prove:
$$\bigcup_{\alpha\in\Gamma} A_\alpha \subseteq \bigcup_{\alpha\in\Lambda} A_\alpha$$

   Proof. Let $x \in \bigcup_{\alpha\in\Gamma} A_\alpha \Rightarrow \exists\, i \in \Gamma$ such that $x \in A_i$, and since $\Gamma \subseteq \Lambda$, $i \in \Lambda$.
   Thus we know $A_i \subseteq \bigcup_{\alpha\in\Lambda} A_\alpha$ and since $x \in A_i$, we have $x \in \bigcup_{\alpha\in\Lambda} A_\alpha$.
   Therefore $\bigcup_{\alpha\in\Gamma} A_\alpha \subseteq \bigcup_{\alpha\in\Lambda} A_\alpha$. □

2. Using mathematical induction show that given any two real numbers a and b, $a - b$ is a factor of $a^n - b^n$ for all $n \in \mathbb{N}$.

   Proof. Using induction on n we have:

   Basis step: For $n = 1$ we have $a - b$ is a factor of $a^1 - b^1$ which is a true statement.

   Inductive step: Assume for $n = k$ that $a - b$ is a factor of $a^k - b^k$ and show this implies for $n = k+1$ that $a - b$ is a factor of $a^{k+1} - b^{k+1}$.
   Consider
$$a^{k+1} - b^{k+1} = a \cdot a^k - b \cdot b^k = a \cdot a^k - a \cdot b^k + a \cdot b^k - b \cdot b^k = a(a^k - b^k) + (a-b)b^k$$
   From our basis step (and in general) we see that $(a - b)$ divides $(a - b)b^k$ and from our inductive step we know that $(a - b)$ divides $a(a^k - b^k)$. Therefore, $(a - b)$ divides $a(a^k - b^k) + (a - b)b^k = a^{k+1} - b^{k+1}$.

   Thus, for all $n \in \mathbb{N}$ $a - b$ is a factor of $a^n - b^n$. □

3. Prove
$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} \geq \sqrt{n} \text{ for all } n \in \mathbb{N}.$$

   Proof. Using induction on n we have:

   Basis step: For $n = 1$ we have $\frac{1}{\sqrt{1}} \geq \sqrt{1}$ which is a true.

   Inductive step: Assume for $n = k$ that
$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{k}} \geq \sqrt{k} \tag{1}$$
   and show that this implies for $n = k + 1$ we have
$$\frac{1}{\sqrt{1}} + \cdots + \frac{1}{\sqrt{k+1}} \geq \sqrt{k+1} \tag{2}$$
   We see that the left hand side of (1) and (2) differ by $\frac{1}{\sqrt{k+1}}$.
   Thus, adding this term to the right hand side of (1) gives
$$\frac{1}{\sqrt{1}} + \cdots + \frac{1}{\sqrt{k+1}} \geq \sqrt{k} + \frac{1}{\sqrt{k+1}} = \frac{\sqrt{k(k+1)} + 1}{\sqrt{k+1}} \geq \frac{\sqrt{k(k+1)}}{\sqrt{k+1}} \geq \frac{\sqrt{k(k+1)}}{\sqrt{k}} = \sqrt{(k+1)}$$
   or, in short,
$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{k+1}} \geq \sqrt{k+1}$$

   Therefore, $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} \geq \sqrt{n}$ for all $n \in \mathbb{N}$. □

4. Let $f : S \to T$ be a function with C and D subsets of T.
Prove: $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$.

Proof. Let $x \in f^{-1}(C \cap D)$ and consider

$$\begin{aligned} x \in f^{-1}(C \cap D) &\Leftrightarrow f(x) \in C \cap D \\ &\Leftrightarrow f(x) \in C \text{ and } f(x) \in D \\ &\Leftrightarrow x \in f^{-1}(C) \text{ and } x \in f^{-1}(D) \\ &\Leftrightarrow x \in f^{-1}(C) \cap f^{-1}(D) \end{aligned}$$

Therefore, $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$. □

5. Let $f : S \to T$ be a function. Prove that $f(A \cap B) = f(A) \cap f(B)$ for all subsets A and B of S if and only if f is an injection.

Proof. We will prove each part of the if and only if separately:

$\Rightarrow$ If $f(A \cap B) = f(A) \cap f(B)$ for all subsets A and B of S then f is an injection.
Let $x, y \in S$ such that $f(x) = f(y) = z$ for $z \in f(A \cap B)$. Also, since our hypothesis is true for all subsets A and B of S, consider the subsets $A = S$ and $B = \{x\}$. Then $A \cap B = \{x\}$ and $f(x) \in f(A \cap B)$. Also, we see that $f(y) \in f(A \cap B)$ since $f(y) = f(x)$. Since $f(A \cap B) = f(A) \cap f(B)$, $f(y) \in f(A) \cap f(B) \Rightarrow f(y) \in f(A)$ and $f(y) \in f(B) \Rightarrow y \in B$. Thus, $y = x$ and we can conclude that f is an injection.

$\Leftarrow$ If f is an injection then $f(A \cap B) = f(A) \cap f(B)$ for all subsets A and B of S.
Let $y \in f(A \cap B)$. Then, since f is an injection, there exists a unique $x \in A \cap B$ such that $f(x) = y$.
$x \in A \cap B \Rightarrow x \in A$ and $x \in B \Rightarrow y = f(x) \in f(A)$ and $y = f(x) \in f(B)$. Thus, $y \in f(A) \cap f(B)$.
Therefore $f(A \cap B) \subseteq f(A) \cap f(B)$.
Now let $y \in f(A) \cap f(B) \Rightarrow y \in f(A)$ and $y \in f(B)$. Since $y \in f(A)$, we know there exists an $x_1 \in A$ such that $f(x_1) = y$. Similarly, $y \in f(B) \Rightarrow$ there exists an $x_2 \in B$ such that $f(x_2) = y$. However, since f is an injection, $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$. Thus, there exists a unique $x \in A \cap B$ such that $f(x) = y \Rightarrow y \in f(A \cap B)$.
Therefore $f(A) \cap f(B) \subseteq f(A \cap B)$
Thus $f(A \cap B) = f(A) \cap f(B)$ for all subsets A and B of S.

□