MATH 348 - Advanced Engineering Mathematics March 4, 2009

Homework 7, Spring 2009 **Due**: March 20, 2009

COMPLEX FOURIER SERIES AND INTRODUCTION TO FOURIER TRANSFORMS

1. Given,

$$f(x) = \begin{cases} \dfrac{2k}{L}x, & 0 < x \le \frac{L}{2} \\ \frac{2k}{L}(L-x), & \frac{L}{2} < x < L \end{cases} \tag{1}$$

   (a) Sketch a graph f on $[-2L, 2L]$.

   (b) Sketch a graph f^*, the even periodically extension of f, on $[-2L, 2L]$.

   (c) Calculate the Fourier cosine series for the half-range expansion of f.

   Comment: Your graphs from part (a) and part (b) should be different outside of $[-L, L]$.

2. In class we derived the complex Fourier series coefficients c_n from the real Fourier series coefficients $a_0,\ a_n,\ b_n$. The coefficients c_n can also be derived using an orthogonality relation similar to the derivations of $a_0,\ a_n,\ b_n$, which is, perhaps, easier.

   (a) First show that $\displaystyle\int_{-\pi}^{\pi} e^{inx}e^{-imx}dx = 2\pi\delta_{nm}$ where $n, m \in \mathbb{Z}$.

   (b) Next using the orthogonality relationship defined in (a), find the Fourier coefficients of $f(x) = \displaystyle\sum_{n=-\infty}^{\infty} c_n e^{inx}$.

   Hint: For part (b) you should first multiply both sides of the equation by e^{-imx}. Next you should integrate from $-\pi$ to π and use the identity $e^{ik\pi} = (-1)^k,\ \ k \in \mathbb{Z}$ to derive c_m. [1]

   Comment: For the complex Fourier series of a $2L$-periodic function you can make a scaling change of variables like we did for the real Fourier series.

3. Let $f(x) = x^2$, $-\pi < x < \pi$, be 2π-periodic.

   (a) Calculate the complex Fourier series representation of f.

   (b) Using the complex Fourier series found in (a), recover the real Fourier series representation of f.

   Hint: For part (b) you will want to follow the example discussed in the class.

4. In differential equations you likely studied the Laplace integral transform, $\displaystyle\int_0^{\infty} f(t)e^{-st}dt$, which describes how similar the function f is to the transform's *kernel*, i.e. an exponential function. If we consider rotating the kernel $\pi/2$ into the complex plane and integrate over all space then we get a Fourier transform $\displaystyle\int_{-\infty}^{\infty} f(t)e^{-ist}dt$. . If $f(t) = (2\pi)^{-\frac{1}{2}}\displaystyle\int_{-\infty}^{\infty}\hat{f}(\omega)e^{-i\omega t}d\omega$ then $\hat{f}(\omega)$ describes the complex amplitude of the sinusoid associated with the angular frequency ω. That is, $|\hat{f}(\omega)|^2 = \hat{f}(\omega)\overline{\hat{f}(\omega)}$ gives the square of the amplitude of the sinusoid for ω and thus is an expression of the power carried by that Fourier mode. In preparation of our study of Fourier transforms read the following wikipedia articles,

   - `http://en.wikipedia.org/wiki/Fourier_transform` - Opening paragraph, introduction and definition.
   - `http://en.wikipedia.org/wiki/Fourier_transform#Cross-correlation_theorem`
   - `http://en.wikipedia.org/wiki/Fourier_transform#Uncertainty_principle`

   and respond to the following,

   (a) How is the Fourier transform related to Fourier series? You should discuss both the periodicity and number of Fourier modes used in the construction of each.

[1] Why is $e^{ik\pi} = (-1)^k$ true?

(b) What does cross-correlation measure? What would auto-correlation measure?

(c) What is the uncertainty principle as it relates to Fourier transforms? How much power would be required to send a signal like $\delta(t)$?

5. If a function is not periodic nor can it be periodically extended then the function has no Fourier series representation. However, in this case the function can have a Fourier integral representation. Analysis of the Fourier integral representation[2] reveals the complex Fourier transform pairs:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} d\omega, \quad \hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx \tag{2}$$

We can connect this transform pait to the complex Fourier series. If a function is periodic then there exists a representation of the function in the countably-infinite basis of imaginary-exponential functions. In this case the coefficients[3] of this expansion are given by an integral whose limits of integration are bounded.[4] These coefficients quantify the amplitude of oscillation for each discrete frequency of oscillation.[5] In the case of f(x), as above, we have that the function f has a representation in the uncountably-infinite basis of imaginary-exponential functions. In this case the coefficients are given by the previous integral whose limits of integration are unbounded.[6] These coefficients quantify the amplitude of oscillation for each continuous frequency of oscillation.

Now, we want to connect all of this to sections 11.7 and 11.8 in our text. We have that the Fourier integral represents functions in the sine/cosine basis without requiring the function to be periodic. Now, what role does symmetry play in this representation? With minimal work we see that if a function is even/odd then the Fourier integral reduces to a Fourier cosine/sine integral.[7] As with our original derivation of transform, if we look at the interplay between the coefficient functions $A(\omega)/B(\omega)$ and $f(x)$ we find the transform pairs:

$$f_c(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \hat{f}(\omega) \cos(\omega x) d\omega \qquad \hat{f}_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos(\omega x) dx \tag{3}$$

$$f_s(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \hat{f}(\omega) \sin(\omega x) d\omega \qquad \hat{f}_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin(\omega x) dx \tag{4}$$

We call (**??**) the Fourier cosine transform pair and, surprisingly, we call (**??**) the Fourier sine transform.

(a) Show that $f_c(x)$ and $\hat{f}_c(\omega)$ are even functions and that $f_s(x)$ and $\hat{f}_s(\omega)$ are odd functions.[8]

(b) Show that if we assume that $f(x)$ is an even function then (**??**) defines the transform pair given by (**??**). Also, show that if $f(x)$ is an odd function then (**??**) defines the transform pair given by (**??**). [9]

Given,

$$f(x) = \begin{cases} A, & 0 < x < a \\ 0, & \text{otherwise} \end{cases}, \qquad A, a \in \mathbb{R}^+. \tag{5}$$

(c) On the same graph plot the even and odd extensions of f.

(d) Find the Fourier cosine and sine transforms of f.

(e) Using the Fourier cosine transform show that $\displaystyle\int_{-\infty}^{\infty} \frac{\sin(\pi\omega)}{\pi\omega} d\omega = 1.$

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[2] See classnotes or Kreyszig pgs. 518-519

[3] also called weights

[4] Recall that our derivation lead to $c_n = \frac{1}{2\pi} \int_{-L}^{L} f(x) e^{i\omega_n x} dx$ where $\omega_n = \frac{n\pi}{L}$.

[5] That is, for each ω_n there is a corresponding c_n where $|c_n|^2$ is a measure of the power of the sinusoids associated with ω_n.

[6] In this case the behavior of f must be known everywhere instead of on the interval $(-L, L)$.

[7] Kreyszig pg. 511

[8] Thus, if an input function has an even or odd symmetry then the transformed function shares the same symmetry.

[9] Thus, if an input function has symmetry then the Fourier transform is real-valued.