MATH332-Linear Algebra **Homework Six Solutions**

Abstract Vector Spaces, Bases and Coordinates, Matrix Spaces

Text: Chapter 4 Section Overviews: 4.1-4.6

Quote of Homework Six Solutions	
	The Adventures of Barron Münchausen : (1988)

1. Abstract Vector Spaces

1.1. **Linear Ordinary Differential Equations.** Verify that the set of all n-times continuously differentiable functions on $[a, b]$, which satisfies the homogeneous linear ordinary differential equation $L[y] = 0$,

$$V = \left\{ y \in C^{(n)}[a,b] : L[y] = a_n(t)\frac{d^n y}{dt^n} + a_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_0(t)y = 0, \text{ where } a_0, \dots, a_n \in C[a,b] \right\},$$

is a vector subspace of the vector space of all functions.[1]

This proof uses the linearity of the derivative. We first note that for $y_1, y_2 \in V$ and $c_1, c_2 \in \mathbb{R}$ we have,

$$\begin{aligned} L[c_1y_1 + c_2y_2] &= \sum_{i=0}^{n} a_n(t)\frac{d^n}{dt^n}[c_1y_1 + c_2y_2] \tag{1}\\ &= c_1\sum_{i=0}^{n} a_n(t)\frac{d^n y_1}{dt^n} + c_2\sum_{i=0}^{n} a_n(t)\frac{d^n y_2}{dt^n} \tag{2}\\ &= c_1L[y_1] + c_2L[y_2] \tag{3}\\ &= c_1 \cdot 0 + c_2 \cdot 0, \tag{4} \end{aligned}$$

which implies that linear combinations of elements in V are again in V. Lastly, we note that $L[0] = 0$ since the derivative of the zero function is zero and the sum of zeros is again zero.

1.2. **Polynomial Subspaces.** Prove that if H is the set of all polynomials up to degree n, such that $p(0) = 0$, then H is a subspace of $\mathbb{P}_n$.

This space is the space of all polynomial functions of degree n that pass through the origin. Clearly, the zero function $p(t) = 0$ has this trait and is trivially a polynomial. Now we must show that if $p_1, p_2 \in H$ then $p(t) = c_1p_1 + c_2p_2 \in H$, for $c_1, c_2 \in \mathbb{R}$. To do this we show that $p(0) = 0$,

$$\begin{aligned} p(0) &= c_1p_1(0) + c_2p_2(0) \tag{5}\\ &= c_1 \cdot 0 + c_2 \cdot 0 = 0, \tag{6} \end{aligned}$$

which completes the proof.

1.3. **Function Subspaces.** Prove that if $H = \{f \in C[a,b] : f(a) = f(b)\}$, then H is a subspace of $C[a,b]$.

This is a space of functions defined on a finite domain and are such that their left endpoint is the same as their right. Clearly, $f(x) = 0$ is such that $f(a) = 0 = f(b)$, which implies that H contains the origin-element. Next, we note that if $f_1, f_2 \in H$ and $c_1, c_2 \in \mathbb{R}$ we have,

$$\begin{aligned} f(a) &= c_1f_1(a) + c_2f_2(a) \tag{7}\\ &= c_1f_1(b) + c_2f_2(b) \tag{8}\\ &= f(b), \tag{9} \end{aligned}$$

which implies H contains its linear combinations and is therefore a vector-subspace.

[1] The critical idea is to show that if $u, v \in V$ then $L[c_1u + c_2v] = 0$ where $c_1, c_2 \in \mathbb{R}$.

2. Matrix Space

Given,

$$\mathbf{A} = \begin{bmatrix} -8 & -2 & -9 \\ 6 & 4 & 8 \\ 4 & 0 & 4 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ -2 & 3 & -3 & -3 & -4 \\ 4 & -6 & 9 & 5 & 9 \\ -2 & 3 & 3 & -4 & 1 \end{bmatrix}. \tag{10}$$

2.1. **Column Space Verification.** Is $\mathbf{w}$ in the column space of $\mathbf{A}$? That is, does $\mathbf{w} \in \text{Col } \mathbf{A}$?

2.2. **Null Space Verification.** Is $\mathbf{w}$ in the null space of $\mathbf{A}$? That is, does $\mathbf{w} \in \text{Nul } \mathbf{A}$?

Recall that the null-space of a matrix is the set of all solutions to $\mathbf{Ax} = \mathbf{0}$. This space tells us about all the points in space the homogeneous linear equations simultaneously intersect. One way to determine if $\mathbf{w}$ is in the null-space of $\mathbf{A}$ is by solving the homogeneous equation and determining if $\mathbf{w}$ is one of these solutions. However, it pays to note that if $\mathbf{w}$ is in the null-space of $\mathbf{A}$ then $\mathbf{Aw} = \mathbf{0}$. A quick check shows,

$$\mathbf{Aw} = \begin{bmatrix} -8 & -2 & -9 \\ 6 & 4 & 8 \\ 4 & 0 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = \mathbf{0}. \tag{11}$$

The column space, on the other hand, is a little different. The column space is the set of all linear combinations of the columns of $\mathbf{A}$. This is also called the spanning set of the columns of $\mathbf{A}$. we have,

$$[\mathbf{A}|\mathbf{w}] \sim \left[\begin{array}{ccc|c} -8 & -2 & -9 & 2 \\ 0 & 20 & 10 & 20 \\ 0 & 0 & 0 & 0 \end{array}\right]. \tag{12}$$

The conclusion is that $\mathbf{w}$ is in both the null-space and column space of $\mathbf{A}$. This is not generally true of a non-trivial vector. In fact, it is never true for rectangular coefficient data.

2.3. **Bases for Nul B.** Determine a basis and the dimension of Nul $\mathbf{B}$.

The following problems will require the use of an echelon form of $\mathbf{B}$. One such form is,

$$\mathbf{B} \sim \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ 0 & 0 & 3 & -1 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \mathbf{C}. \tag{13}$$

The null-space of $\mathbf{B}$ is the set of all solutions to $\mathbf{Ax} = \mathbf{0}$. To find a basis for this space we must explicitly solve the homogeneous equation. Thus, from the echelon form $\mathbf{C}$ we have the following,

$$\begin{aligned}
x_4 &= -3x_5 \\
x_3 &= (x_4 - x_5)/3 = (-3x_5 - x_5)/3 = -\frac{4}{3}x_5 \\
x_1 &= \frac{1}{2}(3x_2 - 6x_3 - 2x_4 - 5x_5) = \frac{1}{2}(3x_2 - 6(-\frac{4}{3}x_5) - 2(-3x_5) - 5x_5) = \\
&= \frac{1}{2}(3x_2 + 8x_5 + 6x_5 - 5x_5) = \frac{3}{2}x_2 + \frac{9}{2}x_5 \\
x_2 &\in \mathbb{R} \\
x_5 &\in \mathbb{R} \\
&\Rightarrow \mathbf{x} = x_2 \begin{bmatrix} 3/2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -9/2 \\ 0 \\ -4/3 \\ -3 \\ 1 \end{bmatrix} \qquad x_2, x_5 \in \mathbb{R}
\end{aligned}$$

Hence, the basis for Nul($\mathbf{A}$) is

$$B_{null} = \left\{ \begin{bmatrix} 3/2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -9/2 \\ 0 \\ -4/3 \\ -3 \\ 1 \end{bmatrix} \right\} \tag{14}$$

and dim(Nul $\mathbf{B}$) = 2. The conclusion is that the five linear four-dimensional objects intersect at many points in $\mathbb{R}^5$. The collection of points forms a two-dimensional subspace, which is spanned by the basis vectors. That is, the linear objects intersect forming a planer subspace of $\mathbb{R}^5$. [2]

2.4. **Bases for Col B.** Determine a basis and the dimension of Col $\mathbf{B}$.
The column-space is the set of all linear combinations of the columns of $\mathbf{B}$. We would like to know a basis for this space, which implies that we must somehow determine the columns of the $\mathbf{B}$ matrix that contain unique directional information. That is, we must find the linearly independent columns of the $\mathbf{B}$ matrix. This information has been made clear through the previous null-space problem. Recall that if a set of vectors is linearly independent then their corresponding homogeneous equation must only have the trivial solution. Since row-reduction does not change the solution to a homogeneous equation we have,

$$\mathbf{Bx} = \mathbf{0} \iff [\mathbf{B}|\mathbf{0}] \sim [\mathbf{C}|\mathbf{0}] \iff \mathbf{Cx} = \mathbf{0}. \tag{15}$$

So, we can see if the columns of $\mathbf{C}$ are linearly independent by considering the linear independence of the columns of $\mathbf{C}$. Clearly, the previous problem shows that the columns of $\mathbf{C}$ are not linearly independent. However, it is also clear from $\mathbf{C}$ that the columns without pivots can be made using the columns with pivots, $\mathbf{c}_2 = -3/2\mathbf{c}_1$ and $\mathbf{c}_5 = 3\mathbf{c}_4 + 4/3\mathbf{c}_3 - (9/2)\mathbf{c}_1$. So, if we take only the pivot columns from $\mathbf{C}$ then we would loose the linearly dependent columns and their free-variables. Consequently, the only solution to $\mathbf{C}_{change} = \mathbf{0}$ would be the trivial solution, which implies the columns are linearly independent.

There is still a problem. While row-reduction did not change the dependence relation, it did change the actual vectors. That is, the column-space of $\mathbf{B}$ is different the the column-space of $\mathbf{C}$. To see this consider the constants necessary for $\mathbf{b}_1 \stackrel{?}{=} k_1\mathbf{c}_1 + k_2\mathbf{c}_3 + k_3\mathbf{c}_4$. [3] So, the conclusion is that we must take the linearly independent columns from $\mathbf{B}$ as told to us by $\mathbf{C}$. Thus, a basis for the column space of $\mathbf{B}$ are the pivot columns of $\mathbf{B}$,

$$B_{ColB} = \left\{ \begin{bmatrix} 2 \\ -2 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} 6 \\ -3 \\ 9 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 5 \\ -4 \end{bmatrix} \right\} \tag{16}$$

and dim(Col$\mathbf{B}$) = 3. The dimension of the column-space is also known as the rank of $\mathbf{B}$. From this we see an example of the so-called rank-nullity theorem, which says that the dimension of the null-space and the dimension of the column-space must always add to be the total number of columns in the matrix. That is,

$$\text{Rank } \mathbf{B} + \dim(\text{Nul}(\mathbf{B}) = n, \text{ where } \mathbf{B} \in \mathbb{R}^{m\times n}.$$

2.5. **Bases for Row B.** Determine a basis and the dimension of Row $\mathbf{B}$.

The row-space of a matrix is the set of all linear combinations of its rows. A basis can be found by taking only the linearly independent rows of the matrix, which can be clearly seen as the non-zero rows of any echelon form. While in the case of a column-space the columns must necessarily come from the original matrix, this is not a requirement for the row-space.[4] Thus, a basis for the row-space of $\mathbf{B}$ is given by,

$$B_{RowB} = \left\{ \begin{matrix} [2\ -3\,6\,2\,5] \\ [0\,0\,3\ -1\,1] \\ [0\,0\,0\,1\,3] \end{matrix} \right\}. \tag{17}$$

[2]It is important to notice how the dimension of the null-space drives the previous statements. I did not draw or try to picture anything.
[3]Answer : There are no constants that allow for this to be true.
[4]The reason for this is that row-operations are linear combinations, $R_i = R_i + \alpha R_j$. Thus using the non-zero rows of any echelon form you can get back to the rows of the original matrix and all linear combinations for that matter.

Since these rows were chosen because of their pivots, the dimension of this space is always equal to the dimension of the column space and $\dim(\text{Row}\mathbf{B}) = \text{Rank } \mathbf{B} = 3$. [5]

3. Theory

Prove the following statements:

3.1. Pivot Review. $\dim \text{Row } \mathbf{A} + \dim \text{Nul } \mathbf{A} = n$ where $\mathbf{A} \in \mathbb{R}^{m \times n}$.

We have that $\dim \text{Row } \mathbf{A} = \text{Rank } \mathbf{A}$. Thus the previous statement is nothing more than the rank-nullity theorem from the text.

3.2. More Pivoting. $\text{Rank } \mathbf{A} + \dim \text{Nul } \mathbf{A}^{\mathrm{T}} = m$ where $\mathbf{A} \in \mathbb{R}^{m \times n}$.

$\dim \text{Null } \mathbf{A}^{\mathrm{T}}$ is the number of columns of $\mathbf{A}^{\mathrm{T}}$ without pivots or the number of rows in $\mathbf{A}$ without pivots. By the previous question Rank $\mathbf{A}$ is the number of pivot rows of $\mathbf{A}$. So, the statement is the number rows without pivots plus the number of rows without must equal the number of rows in $\mathbf{A}$.

3.3. Dimensional Arguments. $\mathbf{Ax}=\mathbf{b}$ has a solution for each $\mathbf{b} \in \mathbb{R}^m$ if and only if the equation $\mathbf{A}^{\mathrm{T}}\mathbf{x} = \mathbf{0}$ has only the trivial solution.[6]

For the forward direction we have that $\mathbf{A}$ has a pivot in each row. This implies that $\mathbf{A}^{\mathrm{T}}$ has a pivot in each column. If $\mathbf{A}^{\mathrm{T}}$ has a pivot in each column then there are no free variables. If there a no free variables then its homogeneous system has only the trivial solution.

For the reverse direction we have that $\mathbf{A}^{\mathrm{T}}\mathbf{x} = \mathbf{0}$ has only the trivial solution implies that $\dim \text{Nul } \mathbf{A}^{\mathrm{T}} = 0$, which by the previous problem implies that the rank of $\mathbf{A}$ is m, which means that $\mathbf{A}$ has a pivot in every row and thus $\mathbf{Ax} = \mathbf{b}$ has a solution for every $\mathbf{b} \in \mathbb{R}^m$.

3.4. Spectral Properties of Transpositions. The characteristic polynomial of $\mathbf{A}$ is equal to the characteristic polynomial of $\mathbf{A}^{\mathrm{T}}$. [7]

By properties of determinants we have, $\det(\mathbf{A} - \lambda \mathbf{I}) = \det(\{\mathbf{A}^{\mathrm{T}} - [\lambda \mathbf{I}]^{\mathrm{T}}\}^{\mathrm{T}}) = \det(\mathbf{A}^{\mathrm{T}} - \lambda \mathbf{I})$, which implies that $\mathbf{A}$ and its transpose share the same characteristic polynomial.

3.5. Invertible Matrix Redux. If $\mathbf{A}$ is an invertible matrix with eigenvalue λ then λ^{-1} is an eigenvalue of $\mathbf{A}^{-1}$.[8]

Let $\mathbf{A}$ be a nonsingular matrix then for an eigenpair $\lambda, \mathbf{x}$ we have,

$$\mathbf{A}^{-1}\mathbf{A}\mathbf{x} = \mathbf{x} \tag{18}$$
$$= \mathbf{A}^{-1}\lambda \mathbf{x}, \tag{19}$$

which implies that $\mathbf{A}^{-1}\mathbf{x} = \lambda^{-1}\mathbf{x}$ and thus $\lambda^{-1}, \mathbf{x}$ is an eigenpair of $\mathbf{A}^{-1}$.

3.6. Invertible Diagonalization. If $\mathbf{A}$ is both diagonalizable and invertible, then so is $\mathbf{A}^{-1}$.[9]

We have that $\mathbf{A} = \mathbf{PDP}^{-1}$ implies $\mathbf{A}^{-1} = (\mathbf{PDP}^{-1}) = PD^{-1}P^{-1}$ for invertible $\mathbf{A}$. By our footnote we have that D^{-1} exists and is diagonal. Thus we have a diagonalization for $\mathbf{A}^{-1}$.

3.7. Transpositions if Diagonalization. If $\mathbf{A}$ has n linearly independent eigenvectors, then so does $\mathbf{A}^{\mathrm{T}}$. [10]

If $\mathbf{A}$ has $n-$many linearly independent vectors then it has a diagonalization whose transpose is $\mathbf{A}^{\mathrm{T}} = (\mathbf{P}^{-1})^{\mathrm{T}}\mathbf{D}^{\mathrm{T}}\mathbf{P}^{\mathrm{T}}$. If we define $\mathbf{Q} = (\mathbf{P}^{-1})^{\mathrm{T}} = (\mathbf{P}^{\mathrm{T}})^{-1}$ then we have that $\mathbf{A}^{\mathrm{T}} = \mathbf{QDQ}^{-1}$ since $\mathbf{D}$ is symmetric. Thus, $\mathbf{A}^{\mathrm{T}}$ has a diagonalization and thus $n-$many linearly independent eigenvectors.

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[5] It is possible to take the corresponding rows from $\mathbf{A}$ but dangerous. The reason why is that the rows of the echelon form may not correspond directly to the rows of the original matrix because of row-swaps. However, if you wanted to take the rows from $\mathbf{B}$ and have kept track of your row-swaps then there shouldn't be a problem.

[6] For the forward direction use theorem 1.4.4 on page 43 and problem 3.3 to prove that the dimension of the null space of $\mathbf{A}^{\mathrm{T}}$ is zero.

[7] Note that $\mathbf{I}$ is a symmetric matrix then use rules for the transposition of a sum and determinants of transposes.

[8] Start with $\mathbf{Ax} = \lambda \mathbf{x}$ and multiply on the left by $\mathbf{A}^{-1}$.

[9] Note that if $\mathbf{D}$ is a diagonal matrix then $\mathbf{D}^{-1}$ is the matrix whose diagonal elements are scalar inverses of the diagonal elements of $\mathbf{D}$.

[10] Use theorem 5.3.5 and the fact that if $\mathbf{P}$ is invertible then $(\mathbf{P}^{\mathrm{T}})^{-1} = (\mathbf{P}^{-1})^{\mathrm{T}}$. It is also useful to note that diagonal matrices are symmetric.

4. Change of Bases

The standard basis for $\mathbb{R}^2$ are the column vectors, $\{\mathbf{e}_1, \mathbf{e}_2\}$ of $\mathbf{I}_{2\times 2}$. In class we looked at the basis $\mathfrak{B} = \{[1,1]^{\mathrm{T}}, [-1,1]^{\mathrm{T}}\}$. This basis is rotated $\frac{\pi}{4}$ radians counter-clockwise from the standard basis and does not preserve the notion of length from the standard coordinate system.

4.1. **Rotations Revisited.** Determine a basis for $\mathbb{R}^2$, which is rotated $\frac{\pi}{4}$ radians counter-clockwise from the standard basis and preserves the unit length associated with the standard basis.

There are multiple ways to think about this:

(1) The linear transformation is changing areas, which is not uncommon. Thus, find a corresponding linear transformation with determinant one.
(2) The columns of the change of coordinate matrix are not of unit length. So, choose the same vectors but normalize them.
(3) These vectors are the standard basis vectors rotated $\pi/4$ degrees in the plane. We have a rotation matrix from a previous homework that will do exactly this.

No matter how you look at it the matrix you should get out of this is,

$$\mathbf{P}_{\mathfrak{B}} = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \tag{20}$$

4.2. **Orthogonal Coordinates.** Show that, for this basis, the change-of-coordinates matrix $\mathbf{P}_{\mathfrak{B}}$ is such that, $\mathbf{P}_{\mathfrak{B}}\mathbf{P}_{\mathfrak{B}}^{\mathrm{T}} = \mathbf{P}_{\mathfrak{B}}^{\mathrm{T}}\mathbf{P}_{\mathfrak{B}} = \mathbf{I}_{2\times 2}$.

This was shown in a previous homework.

4.3. **Coordinate Changes.** Given that $[\mathbf{x}_1]_{\mathfrak{B}} = [\sqrt{2}, \sqrt{2}]^{\mathrm{T}}$ determine $\mathbf{x}_1$ and given that $\mathbf{x}_2 = \left[\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right]^{\mathrm{T}}$ determine $[\mathbf{x}_2]_{\mathfrak{B}}$. Calculate the magnitude of both of the vectors previously calculated.

We have,

$$\mathbf{P}_{\mathfrak{B}}[\mathbf{x}_1]_{\mathfrak{B}} = [1\,1]^{\mathrm{T}} = \mathbf{x}_1 \tag{21}$$

$$\mathbf{P}_{\mathfrak{B}}^{-1}\mathbf{x}_2 = [3\,0]^{\mathrm{T}} = \mathbf{x}_2, \tag{22}$$

and a quick check shows that $\mathbf{x}_1 = [\mathbf{x}_1]_{\mathfrak{B}} = \sqrt{2}$ and $\mathbf{x}_2 = [\mathbf{x}_2]_{\mathfrak{B}} = 3$

5. Polynomial Spaces

The Hermite polynomials are a sequence of orthogonal polynomials, which arise in probability, combinatorics and physics.[11] The first four polynomials in this sequence are given as,

$$H_0(x) = 1, \quad H_1(x) = 2x, \quad H_2(x) = -2 + 4x^2, \quad H_3(x) = -12x + 8x^3, \quad x \in (-\infty, \infty).$$

5.1. **Linear Independence.** Show that $\mathfrak{B} = \{1, 2x, -2 + 4x, -12x + 8x^3\}$ is a basis for $\mathbb{P}_3$.
Hint: Determine the coordinate vectors of the Hermite polynomials relative to the standard basis.
Notice that relative to the standard basis we have,

$$H_0 = [1\ 0\ 0\ 0]^{\mathrm{T}}, \tag{23}$$

$$H_1 = [0\ 2\ 0\ 0]^{\mathrm{T}}, \tag{24}$$

$$H_2 = [-2\ 0\ 4\ 0]^{\mathrm{T}}, \tag{25}$$

$$H_3 = [0\ -12\ 0\ 8]^{\mathrm{T}}, \tag{26}$$

which form a square matrix with determinant 64. Thus the vectors are linearly independent and form a basis for $\mathbb{R}^4$, which is isomorphic to $\mathbb{P}_3$.

[11]In physics these polynomials manifest as the spatial solutions to Schrödinger's wave equation under a harmonic potential, which evolves the probability distribution of a quantum mechanical particle near an energy minimum. As it turns out there are infinitely-many Hermite polynomials and consequently one can show that this particle has infinitely-many allowed quantized energy levels, which are evenly spaced. In probability they arise as different moments of a standard normal distribution.

5.2. **Change of Basis.** Let $\mathbf{p}(x) = 7 - 12x - 8x^2 + 12x^3$. Find the coordinate vector of $\mathbf{p}$ relative to $\mathfrak{B}$.
Hint: Determine $\{c_0, c_1, c_2, c_3\}$ such that $\mathbf{p}(x) = \sum_{i=0}^{3} c_i H_i(\mathrm{x})$.

The previous step defines a change of basis matrix,

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 2 & 0 & -12 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix}, \tag{27}$$

which takes representations under the Hermite basis to the standard basis. That is, $\mathbf{H}\,[\mathbf{p}]_{\mathfrak{B}} = \mathbf{p}$ So, if we are considering the representation of $p(t)$ under the Hermite basis we must calculate,

$$\mathbf{H}^{-1}\mathbf{p} = \begin{bmatrix} 1 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 3/4 \\ 0 & 0 & 1/4 & 0 \\ 0 & 0 & 0 & 1/8 \end{bmatrix} [7\ -12\ -8\ 12]^{\mathrm{T}} = [3\ 3\ -2\ 3/2]^{\mathrm{T}}. \tag{28}$$