From Guenther: "Modern Optics," Chapter 2

POLARIZATION

The displacement of a transverse wave is a vector quantity. We must therefore specify not only the frequency, phase, and direction of the wave but also the magnitude and direction of the displacement. The direction of the displacement vector is called the *direction of polarization* and the plane containing the direction of polarization and the propagation vector is called the *plane of polarization*. This quantity has the same name as the field quantity introduced in **(2-5)**. Because the two terms describe completely different physical phenomena, there should be no danger of confusion.

From our study of Maxwell's equations, we know that **E** and **H**, for a plane wave in free space, are mutually perpendicular and lie in a plane normal to the direction of propagation **k**. We also know that, given one of the two vectors, we can use **(2-17)** to obtain the other. Convention requires that we use the electric vector to label the direction of the electromagnetic wave's polarization. The selection of the electric field is not completely arbitrary. From **(2-29)** and **(2-30)**, we can write the ratio of the forces on a moving charge in an electromagnetic field due to the electric and magnetic fields as

$$\frac{F_E}{F_H} = \frac{eE}{evB}$$

We can replace B, using **(2-19)** to obtain

$$\frac{F_E}{F_H} = \frac{c}{nv} \tag{2-33}$$

where v is the velocity of the moving charge. Assume that a charged particle is traveling in air at the speed of sound so that $v = 335$ m/sec; then the force due to the electric field of a light wave on that particle would be 8.9×10^5 times larger than the force due to the magnetic field. The size of these numbers demonstrates that except in relativistic situations, when $v \approx c$, the interaction of the electromagnetic wave with matter will be dominated by the electric field.

A conventional vector notation is used to describe the polarization of a light wave; however, to visualize the behavior of the electric field vector as light propagates, a geometrical construction is useful. The geometrical construction, called a Lissajous' figure, describes the path followed by the tip of the electric field vector.

Polarization Ellipse

Assume that a plane wave is propagating in the z direction and the electric field, determining the direction of polarization, is oriented in the x, y plane. In complex notation, the plane wave is given by

$$\mathbf{E} = \mathbf{E}_0 e^{i(\omega t - \mathbf{k}\cdot\mathbf{r} + \phi)} = \mathbf{E}_0 e^{i(\omega t - kz + \phi)}$$

This wave can be written in terms of the x and y components of $\mathbf{E}_0$

$$\mathbf{E} = E_{0x} e^{i(\omega t - kz + \phi_1)}\,\hat{\mathbf{i}} + E_{0y} e^{i(\omega t - kz + \phi_2)}\,\hat{\mathbf{j}} \tag{2-34}$$

(We will use only the real part of **E** for manipulation to prevent errors.) We divide each component of the electric field by its maximum value so that the problem is reduced to one of the following two sinusoidally varying unit vectors:

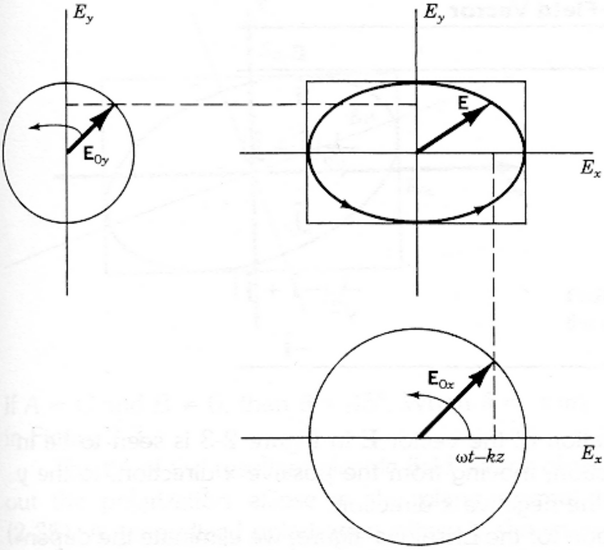

FIGURE 2-3. Geometrical construction showing how the Lissajous' figures are constructed from harmonic motion along the x and y coordinate axes. The harmonic motion along each coordinated axis is created by projecting a vector rotating around a circle onto the axis according to the technique discussed in Appendix 1B-1.

$$\frac{E_x}{E_{0x}} = \cos(\omega t - kz + \phi_1) = \cos(\omega t - kz)\ \cos\phi_1 - \sin(\omega t - kz)\ \sin\phi_1$$

$$\frac{E_y}{E_{0y}} = \cos(\omega t - kz)\ \cos\phi_2 - \sin(\omega t - kz)\ \sin\phi_2$$

When these unit vectors are added together, the result will be a set of figures called *Lissajous' figures*. The geometrical construction shown in Figure 2-3 can be used to visualize the generation of the Lissajous' figure. The harmonic motion along the x axis is found by projecting a vector rotating around a circle of diameter E_{0x} onto the x axis. The harmonic motion along the y axis is generated the same way using a circle of diameter E_{0y}. The resulting x and y components are added to obtain **E**. In Figure 2-3, the two harmonic oscillators both have the same frequency $(\omega t - kz)$, but differ in phase by

$$\delta = \phi_2 - \phi_1 = -\frac{\pi}{2}$$

The tip of the electric field **E** in Figure 2-3 traces out an ellipse, with its axes aligned with the coordinate axes. To determine the direction of the rotation of the vector, assume that $\phi_1 = 0, \phi_2 = -\pi/2$, and $z = 0$ so that

$$\frac{E_x}{E_{0x}} = \cos\ \omega t \qquad \frac{E_y}{E_{0y}} = \sin\ \omega t$$

$$\mathbf{E} = \left(\frac{E_x}{E_{0x}}\right)\hat{\imath} + \left(\frac{E_y}{E_{0y}}\right)\hat{\jmath}$$

The normalized vector **E** can easily be evaluated at a number of values of ωt to discover the direction of rotation. Table 2.3 shows the value of the vector

TABLE 2.3 Rotating E-Field Vector

ωt	$\mathbf{E}$
0	$\hat{\mathbf{i}}$
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}(\hat{\mathbf{i}} + \hat{\mathbf{j}})$
$\frac{\pi}{2}$	$\hat{\mathbf{j}}$
$\frac{3\pi}{4}$	$\frac{1}{\sqrt{2}}(-\hat{\mathbf{i}} + \hat{\mathbf{j}})$
π	$-\hat{\mathbf{i}}$

as ωt increases. The rotation of the vector $\mathbf{E}$ in Figure 2-3 is seen to be in a counterclockwise direction, moving from the positive x direction, to the y direction, and finally to the negative x direction.

To obtain the equation for the Lissajous' figure, we eliminate the dependence of the unit vectors on $(\omega t - kz)$. First, multiply the equations by $\sin\phi_2$ and $\sin\phi_1$, respectively, and then subtract the resulting equations. Second, multiply the two equations by $\cos\phi_2$ and $\cos\phi_1$, respectively, and then subtract the new equations. These two operations yield the following pair of equations:

$$\frac{E_x}{E_{0x}}\sin\phi_2 - \frac{E_y}{E_{0y}}\sin\phi_1 = \cos(\omega t - kz)(\cos\phi_1\sin\phi_2 - \sin\phi_1\cos\phi_2)$$

$$\frac{E_x}{E_{0x}}\cos\phi_2 - \frac{E_y}{E_{0y}}\cos\phi_1 = \sin(\omega t - kz)(\cos\phi_1\sin\phi_2 - \sin\phi_1\cos\phi_2)$$

The term in parens can be simplified using the trig identity

$$\sin\delta = \sin(\phi_2 - \phi_1) = \cos\phi_1\sin\phi_2 - \sin\phi_1\cos\phi_2$$

After replacing the term in parens by $\sin\delta$, the two equations are squared and added, yielding the equation for the Lissajous' figure

$$\left(\frac{E_x}{E_{0x}}\right)^2 + \left(\frac{E_y}{E_{0y}}\right)^2 - \left(\frac{2E_xE_y}{E_{0x}E_{0y}}\right)\cos\delta = \sin^2\delta \qquad (2\text{-}35)$$

The trig identity

$$\cos\delta = \cos(\phi_2 - \phi_1) = \cos\phi_1\cos\phi_2 + \sin\phi_1\sin\phi_2$$

was also used to further simplify **(2-35)**.

Equation **(2-35)** has the same form as the equation of a conic

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

Geometry defines the conic as an ellipse because from **(2-35)**,

$$B^2 - 4AC = \frac{4}{E_{0x}^2E_{0y}^2}(\cos^2\delta - 1) < 0$$

This ellipse is called the *polarization ellipse*. The orientation of the ellipse with respect to the x axis is

$$\tan 2\theta = \frac{B}{A - C} = \frac{2E_{0x}E_{0y}\cos\delta}{E_{0x}^2 - E_{0y}^2} \qquad (2\text{-}36)$$

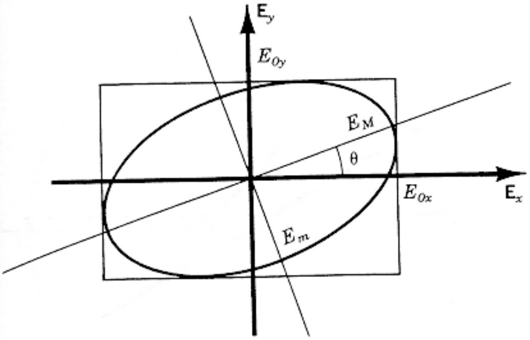

FIGURE 2-4. General form of the ellipse described by **(2-35)**.

If $A = C$ and $B \neq 0$, then $\theta = 45°$. When $\delta = \pm\pi/2$, then $\theta = 0°$ as shown in Figure 2-3.

The tip of the resultant electric field vector obtained from **(2-34)** traces out the polarization ellipse in the plane normal to **k**, as predicted by **(2-35)**. A generalized polarization ellipse is shown in Figure 2-4. The x and y coordinates of the electric field are bounded by $\pm E_{0x}$ and $\pm E_{0y}$. The rectangle in Figure 2-4 illustrates those limits. The component of the electric field along the major axis of the ellipse is

$$E_M = E_x \cos\theta + E_y \sin\theta$$

and along the minor axis of the ellipse is

$$E_m = -E_x \sin\theta + E_y \cos\theta$$

where θ is obtained from **(2-36)**. The ratio of the length of the minor to the major axis of the ellipse is equal to the ellipticity φ, i.e., the amount of deviation of the ellipse from a circle

$$\tan\varphi = \pm\left(\frac{E_m}{E_M}\right) = \frac{E_{0x}\sin\phi_1 \sin\theta - E_{0y}\sin\phi_2\cos\theta}{E_{0x}\cos\phi_1\cos\theta + E_{0y}\cos\phi_2\sin\theta} \tag{2-37}$$

To find the time dependence of the vector **E**, rewrite **(2-34)** in complex form

$$\mathbf{E} = e^{i(\omega t - kz)}(\hat{\mathbf{i}}\, E_{0x} e^{i\phi_1} + \hat{\mathbf{j}}\, E_{0y} e^{i\phi_2}) \tag{2-38}$$

This equation shows explicitly that the electric vector moves about the ellipse in a sinusoidal motion.

By specifying the parameters that characterize the polarization ellipse (θ and φ), we completely characterize a wave's polarization. A review of two special cases will aid in understanding the polarization ellipse.

Linear Polarization

First consider when $\delta = 0$ or π; then **(2-35)** becomes

$$\left(\frac{E_x}{E_{0x}}\right)^2 + \left(\frac{E_y}{E_{0y}}\right)^2 \pm \left(\frac{2E_xE_y}{E_{0x}E_{0y}}\right) = 0$$

The ellipse collapses into a straight line with slope E_{0y}/E_{0x}. The equation of the straight line is

$$\frac{E_x}{E_{0x}} = \pm\frac{E_y}{E_{0y}}$$

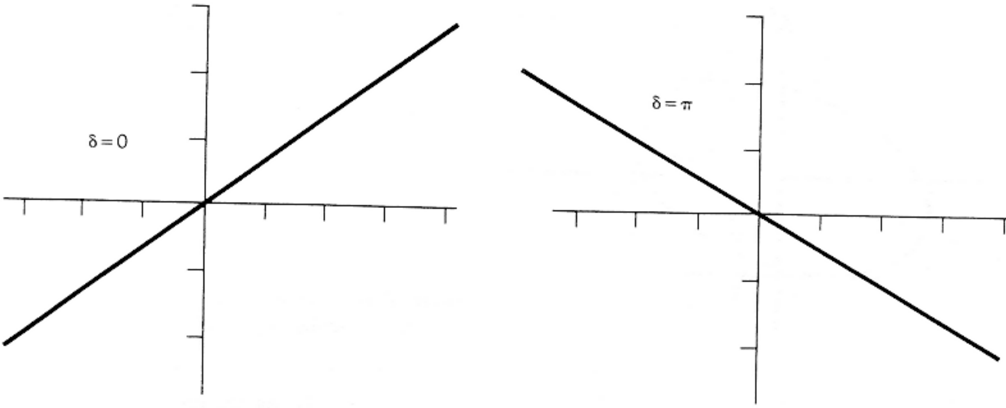

FIGURE 2-5. Lissajous' figures for phase differences between the y and x components of oscillation of 0 and π.

Figure 2-5 displays the straight-line Lissajous' figures for the two phase differences. The θ parameter of the ellipse is the slope of the straight line

$$\tan\ \theta = \frac{E_{0y}}{E_{0x}}$$

resulting in the value of **(2-36)** being given by

$$\tan\ 2\theta = \frac{2\ \tan\ \theta}{1 - \tan^2\ \theta} = \frac{2E_{0x}E_{0y}}{E_{0x}^2 - E_{0y}^2}$$

The φ parameter is given by **(2-37)** as $\tan\ \varphi = 0$.

The time dependence of the **E** vector shown in Figure 2-5 is given by **(2-38)**. The real component is

$$\mathbf{E} = (E_{0x}\hat{\mathbf{i}} \pm E_{0y}\hat{\mathbf{j}})\ \cos(\omega t - kz)$$

At a fixed point in space, the x and y components oscillate in phase (or 180° out of phase) according to the equation

$$\mathbf{E} = (E_{0x}\hat{\mathbf{i}} \pm E_{0y}\hat{\mathbf{j}})\ \cos(\omega t - \phi)$$

The electric vector undergoes simple harmonic motion along the line defined by E_{0x} and E_{0y}. At a fixed time, the electric field varies sinusoidally along the propagation path (the z axis) according to the equation

$$\mathbf{E} = (E_{0x}\hat{\mathbf{i}} \pm E_{0y}\hat{\mathbf{j}})\ \cos(\phi - kz)$$

This light is said to be *linearly polarized.*

Circular Polarization

The second case occurs when $E_{0x} = E_{0y} = E_0$ and $\delta = \pm\pi/2$. From **(2-35)**,

$$\left(\frac{E_x}{E_0}\right)^2 + \left(\frac{E_y}{E_0}\right)^2 = 1$$

The ellipse becomes a circle as shown in Figure 2-6. For this polarization, $\tan\ 2\theta$ is indeterminate and $\tan\ \varphi = 1$.

From **(2-38)**, the temporal behavior is given by

$$\mathbf{E} = E_0[\cos(\omega t - kz)\,\hat{\mathbf{i}} \pm \sin(\omega t - kz)\,\hat{\mathbf{j}}\,]$$

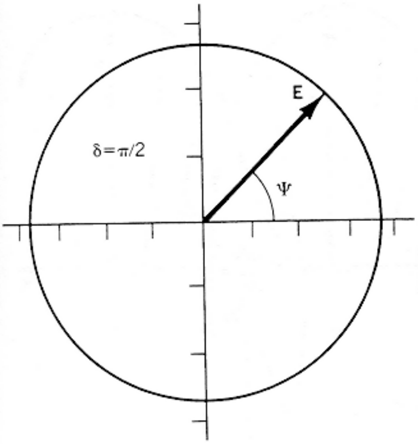

FIGURE 2-6. Lissajous' figures for the case when the phase difference between the y and x components of oscillation differ by $\pm(\pi/2)$ and the amplitudes of the two components are equal. The tip of the electric field vector shown moves along the circle.

The time dependence of the angle Ψ that the **E** field makes with the x axis in Figure 2-6 can be obtained by finding the tangent of the angle Ψ.

$$\tan \Psi = \frac{E_y}{E_x} = \pm \frac{\sin(\omega t - kz)}{\cos(\omega t - kz)} = \pm \tan(\omega t - kz)$$

The interpretation of this result is that at a fixed point in space, the **E** vector rotates in a clockwise direction if $\delta = \pi/2$ and a counterclockwise direction if $\delta = -\pi/2$.

In particle physics, the light would be said to have a negative helicity if it rotated in a clockwise direction. If we look at the source, the electric vector seems to follow the threads of a left-handed screw, agreeing with the nomenclature that left-handed quantities are negative. However, in optics the light that rotates clockwise as we view it traveling toward us from the source is said to be *right-circularly polarized*. The counterclockwise rotating light is *left-circularly polarized*.

The association of right-circularly polarized light with "right handedness" in optics came about by looking at the path of the electric vector in space at a fixed time; then, $\tan \Psi = \tan(\phi - kz)$. See Figure 2-7. As shown in Figure 2-7, right-circular polarized light at a fixed time seems to spiral in a counterclockwise fashion along the z direction, following the threads of a right-handed screw.

This motion can be generalized to include elliptical polarized light when $E_{0x} \neq E_{0y}$. Figure 2-3 schematically displays the generation of the Lissajous' figure for the case of $\delta = \pi/2$, but with unequal values of E_{0x} and E_{0y}. Figure 2-8 shows two calculated Lissajous' figures. If the electric vector moves around the ellipse in a clockwise direction, as we face the source, then the phase difference and ellipticity are

$$0 \leq \delta \leq \pi \qquad \text{and} \qquad 0 < \varphi < \frac{\pi}{4}$$

and the polarization is right-handed. If the motion of the electric vector is moving in a counterclockwise direction, then the phase difference and ellipticity are

$$-\pi \leq \delta \leq 0 \qquad \text{and} \qquad -\frac{\pi}{4} < \varphi < 0$$

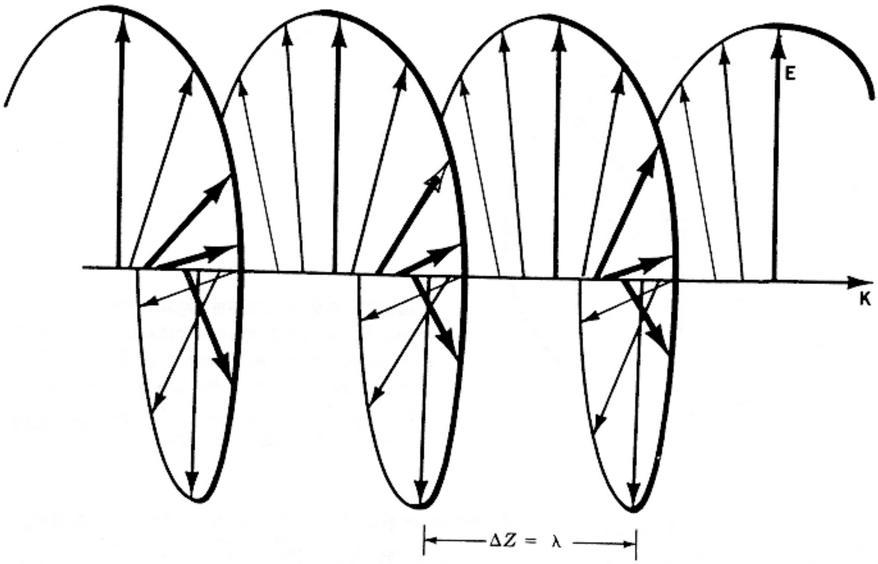

FIGURE 2-7. The path of the electric vector of right-circular polarized light at a fixed time.

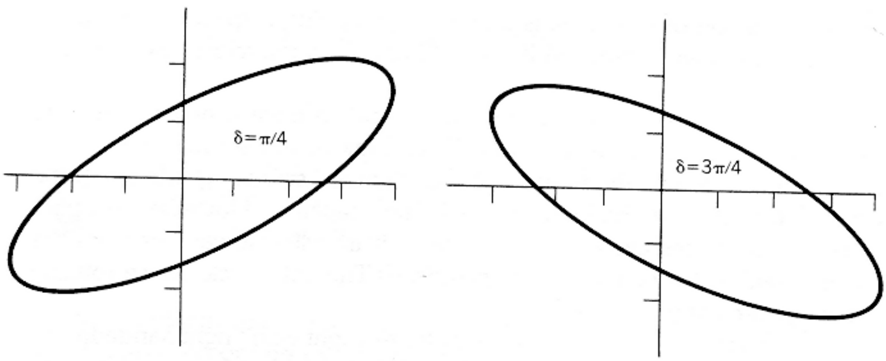

FIGURE 2-8. Lissajous' figures for elliptical polarized light. They were calculated with $E_{0x} = 0.75$ and $E_{0y} = 0.25$.

The orientation of either ellipse with respect to the x axis will be given by **(2-36)** and will depend upon the relative magnitudes of E_{0x} and E_{0y}.

The procedure used to decompose an arbitrary polarization into polarizations parallel to two axes of a Cartesian coordinate system is a technique used extensively in vector algebra to simplify mathematical calculations. According to the mathematical formalism associated with this technique, the polarization is described in terms of a set of basis vectors $\mathbf{e}_i$. An arbitrary polarization would be expressed as

$$\mathbf{E} = \sum_{i=1}^{2} a_i \mathbf{e}_i \qquad (2\text{-}39)$$

The set of basis vectors $\mathbf{e}_i$ is orthonormal, i.e.,

$$\mathbf{e}_i\mathbf{e}_j^* = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

where we have assumed that the basis vectors could be complex. We mention this mathematical formalism because an identical formalism is encountered in elementary particle physics in which it is used to describe spin.[4]

In a Cartesian coordinate system, the $\mathbf{e}_i$'s are the unit vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, $\hat{\mathbf{k}}$. The summation in **(2-39)** extends over only two terms because the electromagnetic wave is transverse, confining **E** to a plane normal to the direction of propagation (according to the coordinate convention we have selected, the **E** field is in the x, y plane).

The polarization could also be described in terms of a right-circularly polarized component

$$\mathbf{E}_{\mathcal{R}} = E_{0\mathcal{R}}\left[\hat{\mathbf{i}}\cos(\omega t - kz) - \hat{\mathbf{j}}\sin(\omega t - kz)\right]$$

and a left-circularly polarized component

$$\mathbf{E}_{\mathcal{L}} = E_{0\mathcal{L}}\left[\hat{\mathbf{i}}\cos(\omega t - kz) + \hat{\mathbf{j}}\sin(\omega t - kz)\right]$$

An arbitrary elliptical polarization would then be written as

$$\begin{aligned} \mathbf{E} &= \mathbf{E}_{\mathcal{R}} + \mathbf{E}_{\mathcal{L}} \\ &= \hat{\mathbf{i}}\,(\mathbf{E}_{0\mathcal{R}} + \mathbf{E}_{0\mathcal{L}})\cos(\omega t - kz) - \hat{\mathbf{j}}\,(\mathbf{E}_{0\mathcal{R}} - \mathbf{E}_{0\mathcal{L}})\sin(\omega t - kz) \qquad (2\text{-}40) \end{aligned}$$

The geometrical construction that demonstrates the expression of an arbitrary elliptical polarized light wave in terms of right and left circularly polarized waves is shown in Figure 2-9. The use of circular polarized waves as the basis set for describing polarization is discussed by Klein.[5]

In the formalism associated with **(2-39)**, the expansion coefficients a_i can be used to form a 2–2 matrix, which in statistical mechanics is called the *density matrix* and in optics the *coherency matrix*.[6] The elements of the matrix are formed by the rule

$$\rho_{ij} = \mathbf{a}_i\mathbf{a}_j^*$$

We will not develop the theory of polarization using the coherency matrix, but simply use the coherency matrix to justify the need for four independent measurements to characterize polarization. There is no unique set of measurements required by theory but normally measurements made are of the *Stokes parameters*, which are directly related to the polarization ellipse of Figure 2-4. (We will see in a few moments that only three of the four measurements are independent. This will be in agreement with the definition of the coherency matrix where $\rho_{ij} = \rho_{ji}^*$, i.e., the matrix is Hermitian.)

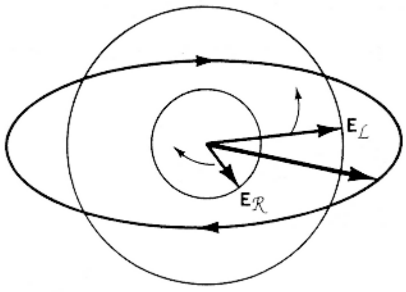

FIGURE 2-9. Construction of elliptical polarized light from two circularly polarized waves.

STOKES PARAMETERS

The Stokes parameters[7] of a light wave are measurable quantities, defined as

$s_0 \rightarrow$ Total flux density.

$s_1 \rightarrow$ Difference between flux density transmitted by a linear polarizer oriented parallel to the x axis and one oriented parallel to the y axis. The x and y axes are usually selected to be parallel to the horizontal and vertical directions in the laboratory.

$s_2 \rightarrow$ Difference between flux density transmitted by a linear polarizer oriented at 45° to the x axis and one oriented at 135°.

$s_3 \rightarrow$ Difference between flux density transmitted by a right-circular polarizer and a left-circular polarizer.

The physical instruments that can be used to measure the Stokes parameters will be discussed in Chapter 13.

If the Stokes parameters are to characterize the polarization of a wave, they must be related to the parameters of the polarization ellipse. It is therefore important to establish that the Stokes parameters are variables of the polarization ellipse **(2-35)**.

In its current form, **(2-35)** contains no measurable quantities and thus must be modified if it is to be associated with the Stokes parameters. In the discussion of the Poynting vector, it was pointed out that the time average of the Poynting vector is the quantity observed when measurements are made of light waves. We must, therefore, find the time average of **(2-35)** if we wish to relate its parameters to observable quantities. To simplify the discussion, assume that the amplitudes of the orthogonally polarized waves, E_{0x} and E_{0y} and their relative phase δ are constants. We will also use the shorthand notation for a time average introduced in **(2-24)**

$$\langle E_x^2 \rangle = \frac{1}{T}\int_{t_0}^{t_0+T} E_{0x}^2\left[\cos(\omega t - kz)\cos\phi_1 - \sin(\omega t - kz)\sin\phi_1\right]^2 dt$$

The time average of **(2-35)** can now be written

$$\frac{\langle E_x^2\rangle}{E_{0x}^2} + \frac{\langle E_y^2\rangle}{E_{0y}^2} - 2\frac{\langle E_x E_y\rangle}{E_{0x}E_{0y}}\cos\delta = \sin^2\delta \qquad (2\text{-}41)$$

Multiplying both sides of **(2-41)** by $(2E_{0x}E_{0y})^2$ removes the terms in the denominators of **(2-41)**

$$4E_{0y}^2\langle E_x^2\rangle + 4E_{0x}^2\langle E_y^2\rangle - 8E_{0x}E_{0y}\langle E_x E_y\rangle\cos\delta = (2E_{0x}E_{0y}\sin\delta)^2$$

The same argument that was used to simplify **(2-25)** can be used to obtain the time averages for the first two terms

$$\langle E_x^2\rangle = \frac{E_{0x}^2}{2}, \qquad \langle E_y^2\rangle = \frac{E_{0y}^2}{2}$$

The calculation of the time average in the third term

$$\langle E_x E_y\rangle = \frac{1}{2}E_{0x}E_{0y}\cos\delta \qquad (2\text{-}42)$$

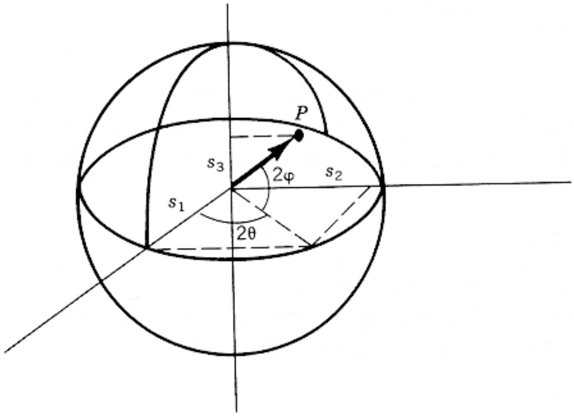

FIGURE 2-10. Poincaré's sphere.

Before it was discovered that the Stokes parameters could be treated as elements of a column matrix, a geometric construction was used to determine the effect of an anisotropic medium on polarized light. The parameters s_1, s_2, s_3 are viewed as the Cartesian coordinates of a point on a sphere of radius s_0. This sphere is called the *Poincaré sphere*[8] and is shown in Figure 2-10.

On the sphere, right-hand polarized light is represented by points on the upper half-surface. Linear polarization is represented by points on the equator. Circular polarization is represented by the poles. With the development of the matrix view of polarization, the usefulness of the Poincaré's sphere has decreased and it is now, for many people, only of historical interest.

is left as a problem, Problem 2-12. With these time averages, **(2-41)** can be written as

$$4E_{0x}^2 E_{0y}^2 - (2E_{0x}E_{0y} \cos \delta)^2 = (2E_{0x}E_{0y} \sin \delta)^2$$

If $E_{0x}^2 + E_{0y}^2$ is added to both sides of this equation, it can be rewritten

$$(E_{0x}^2 + E_{0y}^2)^2 - (E_{0x}^2 - E_{0y}^2)^2 - (2E_{0x}E_{0y} \cos \delta)^2 = (2E_{0x}E_{0y} \sin \delta)^2 \quad \text{(2-43)}$$

Each term in this equation can be identified with a Stokes parameter.

In our derivation, we required that the amplitudes and relative phase of the two orthogonally polarized waves be a constant, but we can relax this requirement and instead define the Stokes parameters as temporal averages. With this modification, the terms of **(2-43)** become

$$s_0 = \langle E_{0x}^2 \rangle + \langle E_{0y}^2 \rangle, \qquad s_1 = \langle E_{0x}^2 \rangle - \langle E_{0y}^2 \rangle$$
$$s_2 = \langle 2E_{0x}E_{0y} \cos \delta \rangle, \qquad s_3 = \langle 2E_{0x}E_{0y} \sin \delta \rangle \quad \text{(2-44)}$$

Equation **(2-43)** can now be written as

$$s_0^2 - s_1^2 - s_2^2 = s_3^2 \quad \text{(2-45)}$$

For a polarized wave, only three of the Stokes parameters are independent. This agrees with the requirement placed on elements of the Hermitian coherency matrix introduced above.

With this demonstration of the connection between the Stokes parameters and the polarization ellipse, the Stokes parameters can be written in terms of the parameters of the polarization ellipse in Figure 2-4.

$$\begin{aligned} s_1 &= s_0 \cos 2\varphi \cos 2\theta \\ s_2 &= s_0 \cos 2\varphi \sin 2\theta \\ s_3 &= s_0 \sin 2\varphi \end{aligned} \quad \text{(2-46)}$$

It is this close relationship between the Stokes parameters and the polarization ellipse that makes the Stokes parameters a useful characterization of polarization.

The Stokes parameters can be used to describe the *degree of polarization* defined as

$$V = \frac{1}{s_0}\sqrt{s_1^2 + s_2^2 + s_3^2} \qquad (2\text{-}47)$$

[The equality of **(2-45)** applies to completely polarized light when $V = 1$.] The degree of polarization can be used to characterize any light source that is physically realizable. If the time averages used in the definition of the Stokes parameters s_2 and s_3 are zero,

$$\langle E_{0x}^2 \rangle = \langle E_{0y}^2 \rangle \qquad \text{and} \qquad s_0 = 2\langle E_{0x}^2 \rangle$$

then the light wave is said to be unpolarized and $V = 0$.

H. Mueller[7] pointed out that the Stokes parameters can be thought of as elements of a column matrix or a 4-vector; see Table 2.4.

$$\begin{pmatrix} s_0 \\ s_1 \\ s_2 \\ s_3 \end{pmatrix}$$

TABLE 2.4 Jones and Stokes Vectors

Jones	Stokes
Horizontal Polarization	
$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$
Vertical Polarization	
$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$
+45° Polarization	
$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$
−45° Polarization	
$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$
Right-Circular Polarization	
$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$
Left-Circular Polarization	
$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$

This view will allow us to follow a polarized wave through a series of optical devices through the use of matrix algebra as we will see later.

JONES VECTOR

There is one other representation of polarized light, complementary to the Stokes parameters, developed by **R. Clark Jones** in 1941 and called the *Jones vector*. It is superior to the Stokes vector in that it handles light of a known phase and amplitude with a reduced number of parameters. It is inferior to the Stokes vector in that, unlike the Stokes representation, that is experimentally determined, the Jones representation cannot handle unpolarized or partially polarized light. The Jones vector is a theoretical construct that can only describe light with a well-defined phase and frequency. The density matrix formalism can be used to correct the shortcomings of the Jones vector, but then the simplicity of the Jones representation is lost.

If we assume that the coordinate system is such that the electromagnetic wave is propagating along the z axis, it was shown earlier that any polarization could be decomposed into two orthogonal **E** vectors, say for this discussion, parallel to the x and y directions. The Jones vector is defined as a two-row, column matrix consisting of the complex components in the x and y direction

$$\mathbf{E} = \begin{bmatrix} E_{0x}\exp\{i(\omega t - \mathbf{k}\bullet\mathbf{r} + \phi_1)\} \\ E_{0y}\exp\{i(\omega t - \mathbf{k}\bullet\mathbf{r} + \phi_2)\} \end{bmatrix} \tag{2-48}$$

If absolute phase is not an issue, then we may normalize the vector by dividing by that number (real or complex) that simplifies the components but keeps the sum of the square of the components equal to 1. For example,

$$\mathbf{E} = \frac{E_{0x}}{\sqrt{E_{0x}^2 + E_{0y}^2}} \exp[i(\omega t - \mathbf{k}\bullet\mathbf{r} + \phi_1)] \begin{bmatrix} 1 \\ \dfrac{E_{0x}}{E_{0y}} e^{i\delta} \end{bmatrix}$$

The normalized vector would be the terms contained within the bracket, each divided by $1/\sqrt{2}$ if $E_{0x} = E_{0y}$. The general form of the Jones vector is

$$\mathbf{E} = \begin{bmatrix} \mathbf{A} \\ \mathbf{B} \end{bmatrix}, \qquad \mathbf{E}^* = \begin{bmatrix} \mathbf{A}^* \ \mathbf{B}^* \end{bmatrix}$$

Some examples of Jones vectors (on the left) and Stokes vectors (on the right) are shown in Table 2.4.