Origin of group velocity
we know that phase velocity is $V_{ph} = \omega/k$
group velocity is $V_g = d\omega/dk$.

expect that the center of a pulse envelope travels
at V_g. $\rightarrow$ $E_o(t) \rightarrow E_o(t - z/V_g)$
Note that t is <u>relative</u> to the pulse peak:
$e^{-t^2/\tau^2} \rightarrow e^{-(t-z/V_g)^2/\tau^2}$ moving peak.

Derive group velocity:
after propagation by L

$$E_{out}(\omega) = \underbrace{E_o \tau \sqrt{\pi}\, e^{-\tau^2(\omega-\omega_0)^2/4}}_{\text{input pulse spectrum}} \underbrace{e^{i\omega n(\omega)L/c}}_{\text{propagation effect}}$$

To describe pulse in the time domain, we must transform back.
- But $e^{i\omega n L/c}$ is complicated
$\therefore$ approximate

Define spectral phase: $\phi(\omega) = \omega n(\omega) L/c$ | L is fixed

$\phi(\omega)$ varies slowly away from resonance

Taylor-expand around ω_0

$$\phi(\omega) \approx \phi(\omega_0) + (\omega-\omega_0)\,\phi'(\omega)\Big|_{\omega_0} + \frac{1}{2!}(\omega-\omega_0)^2\,\phi''(\omega)\Big|_{\omega_0} + O(\Delta\omega^3)$$

$\phi(\omega_0) = \omega_0 n(\omega_0) L/c = k_0 n(\omega_0) L$ = constant phase shift, eval @ ω_0

$\phi'(\omega_0)$ = units of time
$\equiv$ "group delay" $= \frac{d}{d\omega}(kL) = \frac{dk}{d\omega} L = L/V_{gr}$

$\phi''(\omega_0)$ = group delay dispersion.

keep 1st order, transform back to time domain

$$\tilde{E}_{out}(\omega) = \tilde{E}_{in}(\omega)\, e^{ik_o n_o L}\, e^{i(\omega-\omega_o)\phi'(\omega_o)}$$

$$E_{out}(t) = e^{ik_o n_o L}\, \mathcal{F}^{-1}\left\{ \tilde{E}_{in}(\omega)\, e^{i(\omega-\omega_o)\phi'(\omega_o)} \right\}$$

shift theorem:

$$E_{out}(t) = e^{ik_o n_o L}\, E_o \tau\sqrt{\pi}\, \mathcal{F}^{-1}\left\{ e^{-(\omega-\omega_o)^2\tau^2/4}\, e^{i(\omega-\omega_o)\phi'(\omega_o)} \right\}$$

from $\mathcal{F}^{-1}\{F(\omega-\omega_o)\} = f(t)\, e^{-i\omega_o t}$

let $\omega' = \omega - \omega_o$

$$E_{out}(t) = E_o \tau\sqrt{\pi}\, e^{i(k_o n_o L - \omega_o t)}\, \mathcal{F}^{-1}\left\{ e^{-\omega'^2\tau^2/4}\, e^{i\omega'\phi'(\omega_o)} \right\}$$

we know that

$$\mathcal{F}\left\{ e^{-t^2/\tau^2} \right\} = \tau\sqrt{\pi}\, e^{-\omega^2\tau^2/4}$$ transform pair for Gaussian

$$\therefore\ \mathcal{F}^{-1}\left\{ e^{-\omega^2\tau^2/4} \right\} = \frac{1}{\tau\sqrt{\pi}}\, e^{-t^2/\tau^2}$$

also, we use shift theorem on inverse transform:

$$\mathcal{F}^{-1}\left\{ F(\omega)\, e^{i\omega t_o} \right\} = f(t-t_o)$$

here $t_o = \phi'(\omega_o)$ (group delay)

$$E_{out}(t) = E_o\, e^{i(k_o n_o L - \omega_o t)}\, e^{-(t-t_o)^2/\tau^2}$$

$$t_o = \frac{d}{d\omega}(k_o n(\omega))\, L = L/v_g$$

w/ $v_g = d\omega/dk$

output pulse is:

$$E_{out}(t) = E_{in}(t - L/v_g)\, e^{i(k_0 n_0 L)}$$

our coordinate system moves at c

so that w/o dispersion, $E_{out}(t) = E_{in}(t)\, e^{i k_0 n_0 L}$

group delay $\tau_g(\omega) = \phi'(\omega)$ arrival time of a group near ω

note that any information must be communicated with some modulation

→ concept that $v_g < c$

even though $v_{ph} = \omega/k$ can be $> c$

Since τ_g varies w/ ω we will see some broadening of the pulse in addition to shift

- must keep next term in expansion of $\phi(\omega)$ to account for this.

Pulse spreading ("chirp") (hw)

- Expand $\phi(\omega)$ to next order:
- next term is $\propto \frac{1}{2}(\omega-\omega_0)^2\,\phi''(\omega_0)$
- now $\mathfrak{F}^{-1}\left\{ e^{-\Delta\omega^2 \frac{\tau^2}{4}(1-i\Gamma)} \right\}$ use scale thm

result:

> pulse duration is $f(z)$
> $\tau(z) = \tau_0\sqrt{1+\Gamma^2}$

> intensity $\propto 1/\tau(z)$

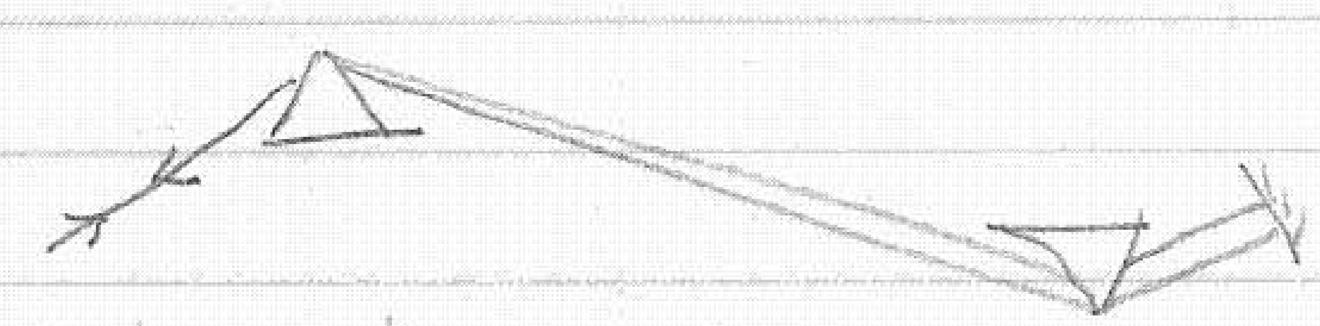
> new phase term: $\phi(t) = -bt^2$ $b \sim \Gamma(z)/\tau^2(z)$ = chirp rate.

total phase: $k_0 n_0 z - \omega_0 t - bt^2$

$-\frac{\partial\phi}{\partial t}$ = instantaneous freq. $= \omega_0 + 2bt$

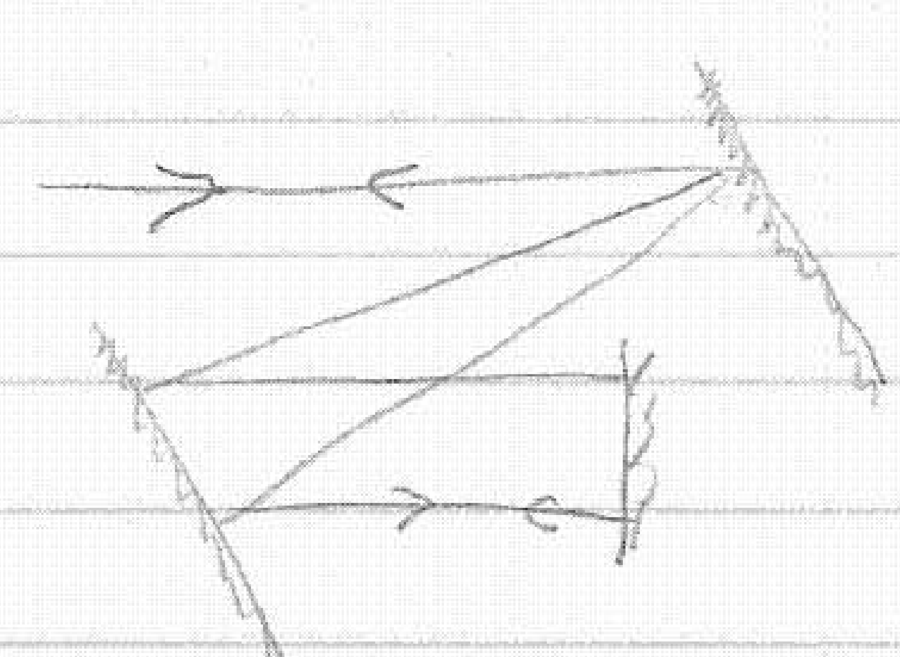
Problem: most materials stretch red first, blue last.
- need a way to compress

1) prisms



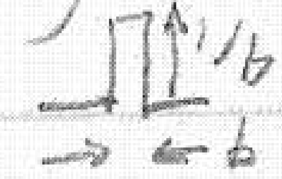
blue → shorter distance than red

2) gratings

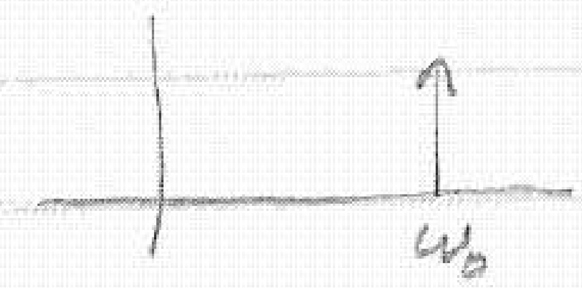


δ-functions:

defined as a limiting function: constant area, ∞ ht.

$\lim b \to 0$ 

$$\mathcal{F}\{e^{-i\omega_0 t}\} = 2\pi\,\delta(\omega-\omega_0)$$



spectrum of pure wave $\to$ right = spike at $+\omega_0$

$\delta(t-t_0)$? short pulse at t_0 $\to$ constant power spectrum!

exercises: $\mathcal{F}(\cos\omega_0 t)$, $\sin\omega_0 t$

$\cos^2\omega_0 t$ $\quad\cos\omega_1 t\cos\omega_2 t$

notation: draw $\delta(\omega-\omega_0)$ w/ unit ht.

$2\delta(\omega-\omega_0)$ as ht of 2

expand trigs as exponentials

Array theorem

consider a regularly-spaced, infinite array of δ-functions.

$$f(t) = \sum_{n=-\infty}^{\infty} \delta(t-nt_0) \equiv \text{III}(t/t_0) \text{ "shah" or "comb" fcn}$$

$$F(\omega) = \sum_{n=-\infty}^{\infty} \mathcal{F}\{\delta(t-nt_0)\}$$

$$= \sum_{n=-\infty}^{\infty} e^{-i\omega t_0 n}$$

$$\to \frac{2\pi}{t_0}\sum_n \delta(\omega - 2\pi n/t_0) = \frac{2\pi}{t_0}\text{III}\left(\omega/(2\pi/t_0)\right)$$

when $\omega t_0 = n\cdot 2\pi$ $\quad e^{in\omega t_0} \to 1$, sum diverges.

$\omega t_0 = n\pi$ $\quad$ " $\to \pm 1 \to 0$ in sum

$\omega t_0 =$ in betw. $\quad$ " random phase $\to$ sum to zero.

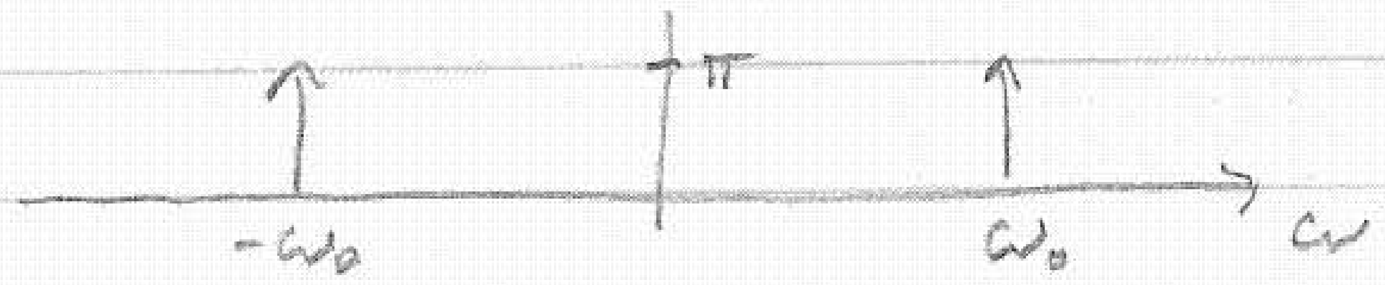
F.T. examples

1) $\mathcal{F}\{\cos\omega_0 t\} = \frac{1}{2}\mathcal{F}\{e^{i\omega_0 t} + e^{-i\omega_0 t}\}$

(real, even)

$$= \frac{1}{2}\left(2\pi\,\delta(\omega-\omega_0) + 2\pi\,\delta(\omega+\omega_0)\right)$$

$$= \pi\,\delta(\omega-\omega_0) + \pi\,\delta(\omega+\omega_0)$$

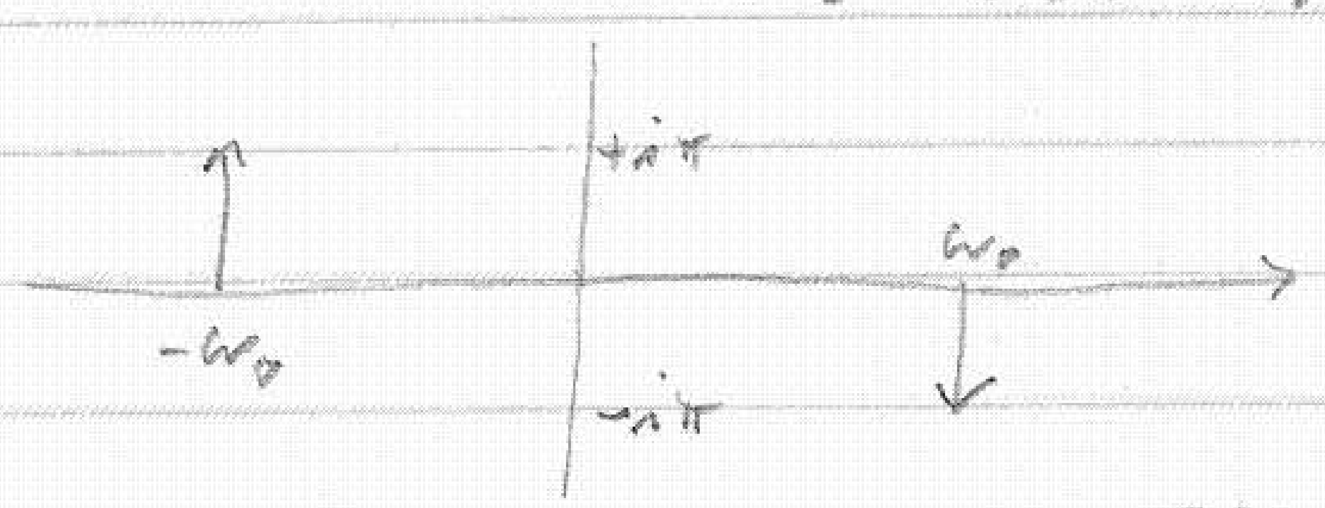


real, even

2) $\mathcal{F}\{\sin\omega_0 t\} = \frac{1}{2i}\cdot 2\pi\left(\delta(\omega-\omega_0) - \delta(\omega+\omega_0)\right)$

(real, odd)

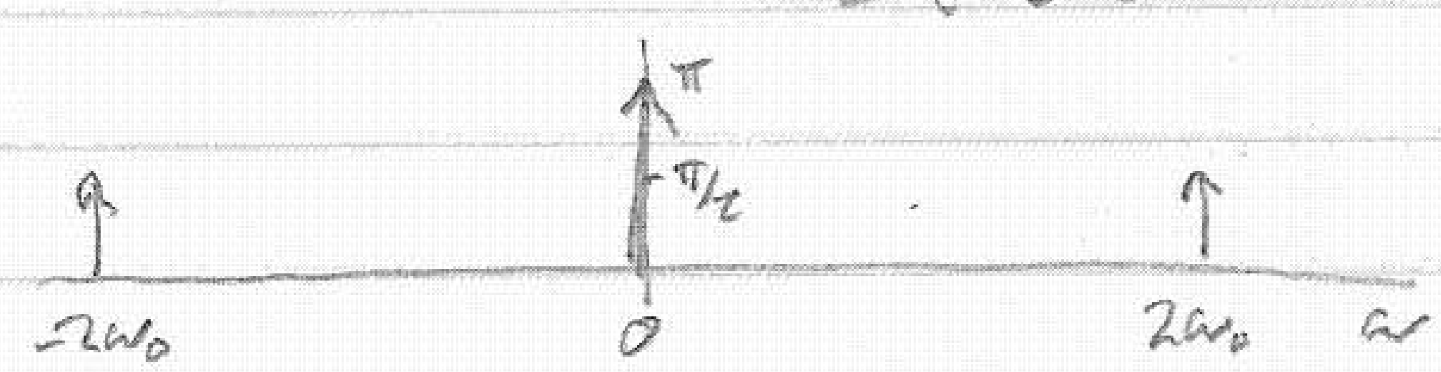
$$= \pi\left(-\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right)$$



imag, odd

3) $\mathcal{F}\{\cos^2\omega_0 t\} = \mathcal{F}\{\frac{1}{4}(e^{2i\omega_0 t} + e^{-2i\omega_0 t} + 2)\}$

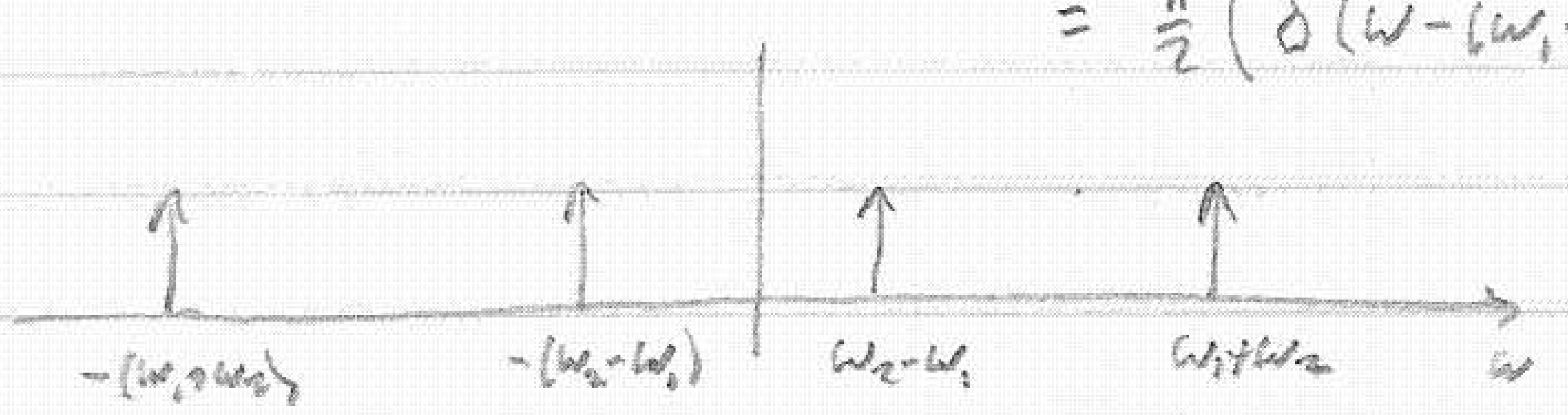
$$= \frac{\pi}{2}\left(\delta(\omega-2\omega_0) + \delta(\omega+2\omega_0) + 2\,\delta(\omega)\right)$$



4) $\mathcal{F}\{\cos\omega_1 t\,\cos\omega_2 t\} = \mathcal{F}\{\frac{1}{4}(e^{i(\omega_1+\omega_2)t} + e^{-i(\omega_1+\omega_2)t} + e^{i(\omega_1-\omega_2)t} + e^{-i(\omega_2-\omega_1)t})\}$

$\omega_2 > \omega_1$

$$= \frac{\pi}{2}\left(\delta(\omega-(\omega_1+\omega_2)) + \delta(\omega+\omega_1+\omega_2) + \delta(\omega-(\omega_1-\omega_2)) + \delta(\omega+(\omega_1-\omega_2))\right)$$



Convolution theorem

define convolution:

$$f(t) \otimes g(t) = \int f(t')\, g(t-t')\, dt'$$

thm: $$= \mathfrak{F}^{-1}\{F(\omega) G(\omega)\}$$

proof:

$$\int \left[\frac{1}{2\pi}\int F(\omega) e^{-i\omega t'} d\omega\right]\left[\frac{1}{2\pi}\int G(\omega') e^{-i\omega'(t-t')} d\omega'\right] dt'$$

$$= \frac{1}{4\pi^2}\int d\omega\, F(\omega) \int d\omega'\, G(\omega') e^{-i\omega' t} \underbrace{\int dt'\, e^{-i(\omega-\omega')t'}}_{2\pi\delta(\omega'-\omega)}$$

now integrate on ω'

$$= \frac{1}{2\pi}\int d\omega\, F(\omega) \int d\omega'\, \delta(\omega'-\omega)\, G(\omega') e^{-i\omega' t}$$

$$= \frac{1}{2\pi}\int d\omega\, F(\omega) G(\omega) e^{-i\omega t} = \mathfrak{F}^{-1}\{F(\omega) G(\omega)\}$$

inverse

$$\mathfrak{F}\{f(t)\, g(t)\} = \frac{1}{2\pi} F(\omega) \otimes G(\omega)$$

ex. square pulse:

$$\mathfrak{F}\{\mathrm{rect}(t/t_0)\, e^{-i\omega_0 t}\} = \frac{1}{2\pi}\left[t_0\, \mathrm{sinc}\left(\frac{\omega t_0}{2}\right)\right] \otimes 2\pi\delta(\omega-\omega_0)$$

$$= t_0\, \mathrm{sinc}\,\frac{(\omega-\omega_0)t_0}{2}$$