MATH348D - October 13, 2011
Exam I - 50 Points

NAME: Key 1

In order to receive full credit, SHOW ALL YOUR WORK. Full credit will be given only if all reasoning and work is provided. When applicable, please enclose your final answers in boxes.

1. (10 Points)

(a) True/False: No Justification Needed

i. The function e^{ix} has no symmetry. T

ii. If a periodic function is neither even nor odd then its Fourier series representation must have sine terms/modes. T

iii. If the complex Fourier coefficients are purely imaginary then the periodic function is even. F

iv. A function can have only one periodic extension. F

v. A truncated Fourier sine half-range expansion will not have Gibb's phenomenon. F

(b) Short Response

i. Provide two physical interpretations of both Fourier coefficients and their corresponding Fourier modes.

Coefficients: i) Intensity
ii) Brightness

Mode: i) Sound tone
ii) Color.

ii. Explain Gibb's phenomenon. What is it and when can you expect it to occur?

Oscillations near a jump discontinuity that occur when a F.S. is truncated

2. (10 Points)

(a) Given that n is an integer for following integrals,

$$\int_{-\pi}^{\pi} f(x)dx = \frac{x^3}{3}\Big|_{-\pi}^{\pi}, \Rightarrow a_0 = \frac{1}{2\pi}\left[\frac{\pi^3}{3} - \frac{-\pi^3}{3}\right] = \frac{\pi^2}{3}$$

$$\int_{-\pi}^{\pi} f(x)\cos(nx)dx = \left[\frac{x^2\sin(nx)}{n} + \frac{2x\cos(nx)}{n^2} - \frac{2\sin(nx)}{n^3}\right]\Big|_{-\pi}^{\pi} = \frac{2\pi(\cos(n\pi))}{n^2} - \frac{-2\pi(\cdots}{n^2}$$

$$\int_{-\pi}^{\pi} f(x)\sin(nx)dx = \left[\frac{x^2\cos(nx)}{n} + \frac{2x\sin(nx)}{n^2} - \frac{2\cos(nx)}{n^3}\right]\Big|_{-\pi}^{\pi} = 0$$

$$\int_{-\pi}^{\pi} g(x)dx = e^{-i\pi} - e^{i\pi}, = (-1)^n - (-1)^n = 0$$

$$\int_{-\pi}^{\pi} g(x)e^{-inx}dx = e^{inx}\left(\frac{1}{n^2} - \frac{ix}{n}\right)\Big|_{-\pi}^{\pi} = (-1)^n\left(\frac{1}{n^2} - \frac{i\pi}{n}\right) - (-1)^n\left(\frac{1}{n^2} + \frac{i\pi}{n}\right)$$

$$= (-1)^{n+1}\frac{i\pi}{n}.2$$

i. Calculate the symmetry and Fourier series of $f(x)$.

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n\cos(nx),$$ Even

ii. Calculate the symmetry and Fourier series of $g(x)$.

$$g(x) = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{(-1)^{n+1}}{n} i e^{inx},$$ odd.

$c_n = a_n \pm i b_n$

$\Rightarrow a_n = 0$ in this case

iii. From the complex Fourier series of g calculate the corresponding real Fourier series.

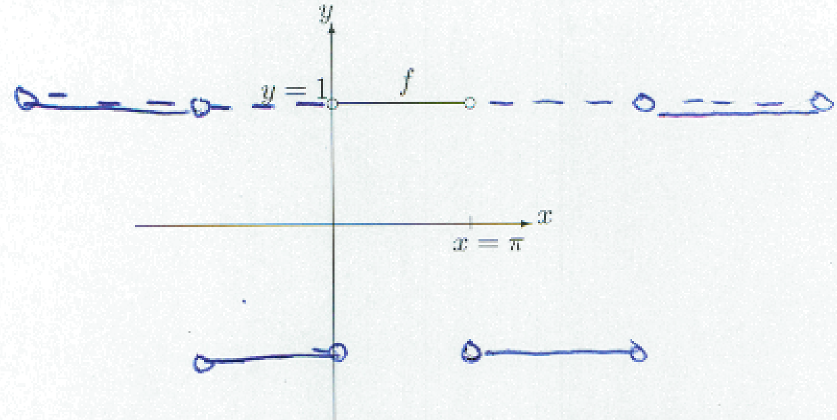
$$g(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} i e^{inx} + \sum_{n=-\infty}^{-1} \frac{(-1)^{n+1}}{n} i e^{inx} =$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} i \left[\underbrace{e^{inx} - e^{-inx}}_{2i\sin(nx)}\right] = \sum_{n=1}^{\infty} \frac{2(-1)^n}{n}\sin(nx)$$

(b) The following table contains different boundary conditions for the ODE, $F'' + \lambda F = 0$, $\lambda \in [0, \infty)$. Fill in each table element with either a yes or a no.

	Boundary value problem has a cosine solution	Boundary value problem has a sine solution	Boundary value problem has a nontrivial constant solution
$F(0) = 0,\ F'(L) = 0$	No	Yes	No
$F'(0) = 0,\ F'(L) = 0$	Yes	No	Yes
$F(0) = 0,\ F(L) = 0$	No	Yes	No
$F'(0) = 0,\ F(L) = 0$	Yes	No	No

3. (10 Points) Suppose f is given by the graph below.



(a) On the graph above, sketch the Fourier cosine and sine half-range expansions with dashed lines and solid lines, respectively.

(b) Find the Fourier coefficients of the cosine and sine half-range expansions. Justify your calculations.

Cosine Half Range: $f_e^*(x) = 1 \Rightarrow a_n = 0, \; a_0 = 1$

Sine Half Range:

$$b_n = \frac{2}{L}\int_0^L f(x)\sin(nx)\,dx = \frac{2}{\pi}\left[-\frac{\cos(nx)}{n}\Big|_0^\pi\right] =$$

$$= -\frac{2}{\pi}\left[(-1)^n - 1\right]$$

4. (10 Points) Find the complex Fourier series representation of

$$f(x) = \begin{cases} 0, & -1 < x < 0, \\ x, & 0 < x < 1. \end{cases}$$

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{\frac{in\pi}{L}x}$$

$$C_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-\frac{in\pi}{L}x}\,dx = \frac{1}{2L}\int_0^1 x e^{-\frac{in\pi}{L}x}\,dx$$

u	dv
x	$e^{-\frac{in\pi}{L}x}$
1	$\frac{Li}{n\pi}e^{-\frac{in\pi}{L}x}$
0	$-\frac{L^2}{n^2\pi^2}e^{-\frac{in\pi}{L}x}$

$$= \frac{1}{2}\left[\frac{Lix}{n\pi}e^{-\frac{in\pi}{L}x}\Big|_0^1 + \frac{L^2}{n^2\pi^2}e^{-\frac{in\pi}{L}x}\Big|_0^1\right] =$$

$$= \frac{1}{2}\left[\frac{i}{n\pi}(-1)^n + \frac{L^2}{n^2\pi^2}(-1)^n - \frac{L^2}{n^2\pi^2}\right], \quad n=0$$

$$C_0 = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$$

5. (10 Points) Given,

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad \begin{array}{l} x \in (0, \pi) \\ t \in (0, \infty) \end{array}, \tag{1}$$

$$u_x(0,t) = 0, \; u_x(\pi, t) = 0, \tag{2}$$

$$u(x, 0) = f(x). \tag{3}$$

(a) Describe the separation of variable procedure used to solve the partial differential equation given by (??)-(??). Be sure to discuss how each step corresponds to each equation (??), (??) and (??).

Step 1: Assume $u(x,t) = F(x)G(t)$

$\Rightarrow$ (1) gives odes

Step 2: Apply (2) to F ode and solve

BVP + time ODE

Step 3: Use superposition + (3) to find unknow constats with F.S.

(b) The separation of variables process, applied to (??), yields the equation

$$F''(x) + \lambda F(x) = 0, \quad \lambda \in \mathbb{R}. \tag{4}$$

Find all nontrivial solutions of (??) that satisfies (??). Justify your choices.

$(2) \Rightarrow \underbrace{F'(0) = 0}, \; F'(L) = 0$

$\hookrightarrow F(x) = \cos(\sqrt{\lambda}\, x)$

$F'(L) = \overset{-\sqrt{\lambda}\sin}{\cancel{\cos}}(\sqrt{\lambda}\, L) = 0 \Rightarrow \sqrt{\lambda_n} = \frac{n\pi}{L}, \; n = 0, 1, 2, 3 \ldots$

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Exam I - 50 Points

NAME: Key 2

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1. (10 Points)

   (a) True/False: No Justification Needed

      i. The function e^{ix} has even symmetry.

      F

      ii. If a periodic function is neither even nor odd then its Fourier series representation must have sine terms/modes.

      T

      iii. If the complex Fourier coefficients are purely real then the periodic function is even.

      T

      iv. A function can have only one periodic extension.

      F

      v. A truncated Fourier sine half-range expansion will not have Gibb's phenomenon.

      F

   (b) Short Response

      i. Provide two physical interpretations of both Fourier coefficients and their corresponding Fourier modes.

      See Key 1

      ii. Explain Gibb's phenomenon. What is it and when can you expect it to occur?

      See Key 1

2. (10 Points)

(a) Given that n is an integer for following integrals,

$$\int_{-\pi}^{\pi} f(x)dx = \frac{x^3}{3}\bigg|_{-\pi}^{\pi},$$

$$\int_{-\pi}^{\pi} f(x)\cos(nx)dx = \left[\frac{x^2\sin(nx)}{n} + \frac{2x\cos(nx)}{n^2} - \frac{2\sin(nx)}{n^3}\right]\bigg|_{-\pi}^{\pi},$$

$$\int_{-\pi}^{\pi} f(x)\sin(nx)dx = \left[\frac{x^2\cos(nx)}{n} + \frac{2x\sin(nx)}{n^2} - \frac{2\cos(nx)}{n^3}\right]\bigg|_{-\pi}^{\pi},$$

$$\int_{-\pi}^{\pi} g(x)dx = e^{-i\pi} - e^{i\pi},$$

$$\int_{-\pi}^{\pi} g(x)e^{-inx}dx = e^{inx}\left(\frac{1}{n^2} - \frac{ix}{n}\right)\bigg|_{-\pi}^{\pi}.$$

i. Calculate the symmetry and Fourier series of $f(x)$.

See Key 1

ii. Calculate the symmetry and Fourier series of $g(x)$.

Key 1

iii. From the complex Fourier series of g calculate the corresponding real Fourier series.

Key 1

(b) The following table contains different boundary conditions for the ODE, $F'' + \lambda F = 0$, $\lambda \in [0, \infty)$. Fill in each table element with either a yes or a no.

	Boundary value problem has a cosine solution	Boundary value problem has a sine solution	Boundary value problem has a nontrivial constant solution
$F(0) = 0,\ F(L) = 0$	No	Yes	No
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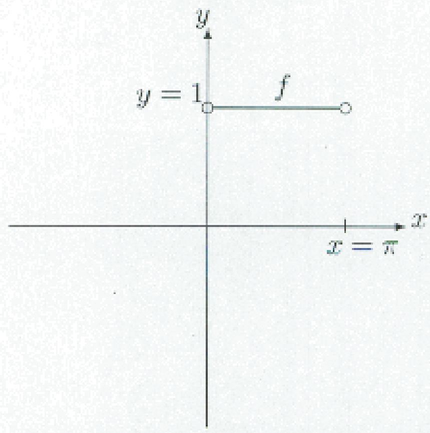
$F'(0) = 0,\ F'(L) = 0$	Yes	No	Yes
$F'(0) = 0,\ F(L) = 0$	Yes	No	No

3. (10 Points) Suppose f is given by the graph below.



(a) On the graph above, sketch the Fourier cosine <u>and</u> sine half-range expansions with dashed lines and solid lines, respectively.

(b) Find the Fourier coefficients of the cosine and sine half-range expansions. Justify your calculations.

See Key 1

4. (10 Points) Find the complex Fourier series representation of

$$f(x) = \begin{cases} x, & -1 < x < 0, \\ 0, & 0 < x < 1. \end{cases}$$

See Key 1 and note

5. (10 Points) Given,

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad \begin{array}{l} x \in (0, \pi) \\ t \in (0, \infty) \end{array}, \tag{1}$$

$$u_x(0,t) = 0,\ u_x(\pi, t) = 0, \tag{2}$$

$$u(x,0) = f(x). \tag{3}$$

(a) Describe the separation of variable procedure used to solve the partial differential equation given by (1)-(3). Be sure to discuss how each step corresponds to each equation (1), (2) and (3).

See Key 1

(b) The separation of variables process, applied to (1), yields the equation

$$F''(x) + \lambda F(x) = 0, \quad \lambda \in \mathbb{R}. \tag{4}$$

Find all nontrivial solutions of (4) that satisfies (2). Justify your choices.

See Key 1