MATH235: Differential Equations with Honors

Second Order Linear Equations

Quote of Second Order Linear Equations Homework	
When I get to the bottom I go back to the top of the slide. Where I stop and turn and I go for a ride 'till I get to the bottom and I see you again.	
	The Beatles : The White Album (1968)

0. Introduction

The following homework concentrates on the second order linear equation,

$$a(t)\frac{d^2y}{dt^2} + b(t)\frac{dy}{dt} + c(t)y = f(t), \tag{1}$$

where a, b, c are smooth functions of time and f is a piecewise smooth or distributional in nature.[1] From class we understand that the general solution to such an equation is given by

$$y(t) = y_h(t) + y_p(t), \tag{2}$$

where $y_h(t)$ is the general solution to the corresponding homogeneous problem of (1) and y_p is exactly one particular solution to (1). Moreover, we know that (1), where $f(t) \equiv 0$ defines a two-dimensional solution space, which implies that we search for homogeneous solutions of the form,

$$y_h(t) = c_1 y_1(t) + c_2 y_2(t), \quad c_1, c_2 \in \mathbb{R}, \tag{3}$$

where y_1, y_2 are both solutions to the homogeneous ODE. Also, we understand that if you know one homogeneous solution to (1) then a second linearly independent solution can be found via

$$y_2(t) = k(t)y_1(t),\ k(t) = \int \frac{p(t)}{[y_1(t)]^2}dt,\ p(t) = e^{-\int (b(t)/a(t))dt}, \tag{4}$$

which gives the general homogeneous solution. Once this is known we can then find the particular solution via

$$y_p(t) = y_2 \int \frac{f(t)y_1(t)}{a(t)W(t)}dt - y_1 \int \frac{f(t)y_2(t)}{atx)W(t)}dt,\ W(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t), \tag{5}$$

That being said, I need to emphasize that this is *the hard way* and that these formulae should only be used as a last resort. I say this because it is always possible to verify proposed solutions to differential equations by direct substitution. This means that if there exists a regimented system of guessing then we can exploit this to find solutions without resorting to (4) and (5). Just as before, we motivate (1) with some mathematical models stemming from Newton's second law.[2]

[1] We won't need the idea of a *distribution* quite yet and when we get there we won't need any theory. So, just let that word float around for a while.

[2] A body of mass m subject to a net force F undergoes an acceleration a that has the same direction as the force and a magnitude that is directly proportional to the force and inversely proportional to the mass, i.e., $F = ma$. Alternatively, the total force applied on a body is equal to the time derivative of linear momentum of the body.

Quote of Linear v. Nonlinear	
Professor Hubert Farnsworth: Good news, everyone. You'll be making a delivery to the planet Trisol. A mysterious planet located in the mysterious depths of the Forbidden Zone. **Leela**: Professor, are we even allowed in the Forbidden Zone? **Professor Hubert Farnsworth**: Why of course. It's just a name, like the Death Zone, or the Zone of No Return. All the zones have names like that in the Galaxy of Terror.	
	Futurama: My Three Suns (1999)

1. Nonlinear Oscillations and Connections to Linear Oscillations

In class we presented a derivation of the linear mass-spring equation. The critical assumptions were

(1) The force due to friction is proportional to the velocity. [3]
(2) The force due to the spring is proportional to the displacement of the mass from the mass-spring equilibrium state.[4]

For now we won't abandon these assumptions but we will try to achieve nonlinearity another way.

1.1. Oscillating Pendulum.

Consider a mass, m, attached to one end of a rigid, but weightless, rod of length L. The other end of the rod is supported at the frictionless origin O, and the rod is free to rotate in the plane. The position of the pendulum is described by the angle, θ, between the rod and the downward vertical direction, with the counterclockwise direction taken as positive.

1.1.1. *Pendulum Equation.* Using the previous description, draw a diagram of the system and by appealing to Newton's second law show that the equation of motion associated with the motion on θ is given by

$$\frac{d^2\theta}{dt^2} + \gamma\frac{d\theta}{dt} + \omega^2 \sin(\theta) = f(t), \tag{6}$$

where the coefficient of kinetic friction is given by $c = m\gamma$, $\omega^2 = g/L$ and f is related to an arbitrary external force. Classify this differential equation by type, order and linearity.

1.1.2. *Small Angle Approximation.* There is no way to solve (6) generally, which is typical of higher order nonlinear equations. However, we can apply to so-called *small angle approximation.* To do this rewrite (6) replacing $\sin(\theta)$ with its Taylor series. If θ is less than one then the higher order terms decrease rapidly and can be discarded. The highest power kept is called the *order* of the approximation. Write down the ODE associated with first-order approximation to the sine function. What equation is this? Can it be solved?

1.1.3. *Nonlinear Effects: Conservative Case.* So, what can be done when we keep either some or all of the nonlinear terms? Well, not a lot generally. However, if we turn off the viscosity/dampening and any external forces then we expect that any energy we put into the pendulum stays in the pendulum for all time. That is, we expect that the system is conservative. Write down the undamped and unforced version of (6) and, using the energy method described in class, derive the conservation law associated with the ODE.

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[3] As we saw earlier in the semester, this assumption can be questioned and in more sophisticated fluid models we appeal to Lord Rayleigh's drag equation, which is quadratic in the velocity variable. See: `http://en.wikipedia.org/wiki/Drag_equation`

[4] Just as with friction, the assumption can be replaced in favor of a more precise model. Doing so essentially asks whether the spring is in the elastic regime, `http://en.wikipedia.org/wiki/Elastic_limit`, which is characterized by Hooke's law, `http://en.wikipedia.org/wiki/Hooke's_law`, where strain is directly proportional to stress. Outside of this limit or for springs in more exotic situations, Hook's law is not a good approximation and nonlinear effects must be taken into account.

1.2. **Undamped, Unforced Pendulum: Solutions.** It is possible to tease a bit of information out of the undamped, unforced pendulum equation,

$$\ddot{\theta} + \omega^2 \sin(\theta) = 0. \tag{7}$$

The starting point is its associated conservation law,

$$\frac{\dot{\theta}^2}{2} - \omega^2 \cos(\theta) = E \in \mathbb{R}, \tag{8}$$

which is a first-order separable ODE.

1.2.1. *Separation of Variables I: Elliptic Integrals.* Using separation of variables find $\theta(t)$.[5]

1.2.2. *Separation of Variables II: Small Angle Approximation.* Assume $\theta \ll 1$ and, using the quadratic approximation to cosine, find $\theta_2(t)$, which solves the second order approximation to the ODE.

1.2.3. *Separation of Variables III: Nonlinear Effects, Fail.* Assume $\theta^2 \ll 1$ and, using the quartic approximation to cosine, find $\theta_4(t)$, which solves the fourth order approximation to the ODE.[6]

Quote of LRC Circuits	
Homer Simpson: A big mountain of sugar is too much for one man. I can see now why God portions it out in those little packets.	
	The Simpsons: Lisa's Rival, S06E02 (1994)

2. Oscillations and LRC Circuits

For this problem we develop a correspondence between our mass-spring oscillations and circuits that have inductors, resistors and capacitors in series. Such a circuit is called and LRC circuit and will behave very much like a mass-spring system. For more information see:

- General Information: `http://en.wikipedia.org/wiki/RLC_circuit`
- Circuit Diagram: `http://en.wikipedia.org/wiki/File:RLC_series_circuit.png`
- LRC in Series: `http://en.wikipedia.org/wiki/RLC_circuit#Series_RLC_circuit`

2.1. **LRC Model.** In reference to the circuit diagram we mention that the current I, measured in amperes, is a function of time t. The resistance R (ohms), the capacitance C (farads), and the inductance L (henrys) are all positive and assumed to be known. The impressed voltage V (volts) is a given function of time. We will also need to understand the total charge Q (coulombs) on the capacitor at time t. The relation between charge Q and current I is $I = Q'$.

To arrive at the model equation we apply Kirchhoff's second law, which states that in a closed circuit the impressed voltage is equal to the sum of the voltage drops in the rest of the circuit. Another way of saying this is that the sum of potential voltage differences across a closed circuit must be zero. [7] Now, we need only understand the voltage drops across the components. To this end we have:

- A resistor will slow the current according to: $I = v_R/R$
- A capacitor will store up charge and create a voltage difference according to: $v_C = Q/C$
- An inductor will create a voltage drop proportional to the time rate of change of the current according to: $v_L = LI'$

2.1.1. *Model Equation.* Using Kirchhoff's second law and the known information relating voltage drops to the circuit elements, derive the model equation for current I in the circuit as a function of time.

2.1.2. *Undriven LRC.* Assume that $V(t) = 0$ for all time and solve the differential equation $LI'' + RI' + C^{-1}I = V(t)$ for $I(t)$ and show that in the presence of resistance the current is a decaying function of time.

[5] This case has been studied and can be managed through the so-called elliptic integrals. See: `http://en.wikipedia.org/wiki/Elliptic_integral`

[6] Actually, you will just want to write down the integral. I can't solve it by hand. :(In fact, I tried a CAS system and it didn't much like it either. The moral here is that nonlinear terms are bad. Update! I'm still working on it and I think there is hope. Extra credit for those who want to journal some ideas about it. Ideas concluding in a solution will be pinned to my fridge.

[7] This is similar to the idea that the line integral for a loop in a conservative field must be zero.

2.1.3. *Driven Vibrations.* Assume that $V(t) = F_0 \cos(\omega_0 t),\ \omega_0 \in \mathbb{R}$ and solve for $I(t)$.

2.1.4. *Phase Angle.* The homogeneous solution is often called the transient solution since it decays in time. The particular solution is often called the steady-state solution since it persists. It is common to report the stead-state solution as one shifted trigonometric function containing a so-called phase angle. Specifically, show that $I_p(t)$ can be written as $I_p(t) = A\cos(\omega_0 t - \delta)$ where

$$A = \frac{F_0}{\Delta} \tag{9}$$

$$\cos(\delta) = \frac{L(\omega^2 - \omega_0^2)}{\Delta} \tag{10}$$

$$\sin(\delta) = \frac{R\omega_0}{\Delta} \tag{11}$$

$$\Delta = \sqrt{L^2(\omega^2 - \omega_0^2)^2 + R^2\omega_0^2} \tag{12}$$

$$\omega^2 = \frac{1}{LC} \tag{13}$$

This allows one to understand the steady-state response in terms of an amplitude A at the cost of a cosine shifted be δ, which is called the phase angle. [8]

2.1.5. *Steady-State Analysis.* What is A for $\omega_0 \to 0$? What is A for $\omega_0 \to \infty$?[9]

2.1.6. *Maximum Steady-State Amplitude.* Find the value of ω_0 for which the amplitude A is maximum. Find A at this value. What occurs to this amplitude when $R \to 0$?

Quote of Laplace Transforms	
Unicron: This is my command. You are to destroy the Autobot matrix of leadership. It is the one thing, the only thing, that stands in my way.	
	Transformers: The Movie (1986)

3. Laplace Transforms

Another way to analyze a linear differential equation, often with constant coefficients, is to use an integral transform. For ordinary differential equations, we use the Laplace transform pair,

$$\mathcal{L}\{f\} = F(s) = \int_0^\infty f(t)e^{-st}dt, \quad \mathcal{L}^{-1}\{F\} = f(t) = \frac{1}{2\pi i}\oint_C F(s)e^{st}ds \tag{14}$$

where C is a contour in the complex plane that avoids all singularities of $F(s)$. The basic idea is that by using an integral it is possible to 'transform away' all of the derivatives in a differential equation, which turns a problem of calculus into a problem of algebra. Since problems of algebra are 'easy,' it is possible to quickly solve the differential equation in the transformed domain. However, this leaves the user with the problem of transforming back to the original domain. Equation (14) shows us that inverse transformation is not easy. However, it is possible to use Eq. (14) to pair functions together so that *formal* inverse transformations can be avoided.

Before we continue, it makes sense to ask why we would care to analyze problems in this way when we have methods, outlined above, that satisfy our needs. Well, one reason is that our guessing methods require $f(t)$ to be of a particular form.[10] In this way the method of undetermined coefficients is rather delicate. However, the Laplace transform is robust. That is, a great deal of functions are Laplace transformable

[8] The idea is that we know the response is going to be trigonometric in nature and the δ is just an offset on where to begin the oscillations. The point is that we are most concerned about amplitude.

[9] This will show that for low frequency forcing the amplitude has a nonzero finite value and that for high frequency forcing the amplitude tends to zero. This now begs the question, is there a frequency between such that the amplitude achieves a maximum?

[10] Specifically, we are talking about using the method of undetermined coefficients, which requires the nonhomogeneous term to be sums or products of exponential and polynomial functions.

and this allows us to solve ODE with more general nonhomogeneous terms. Specifically, we can define the so-called Dirac delta function,

$$\delta_c(t) = \delta(t-c) = 0, \quad t \neq c \in \mathbb{R}, \tag{15}$$

$$\int_{c-\epsilon}^{c+\epsilon} \delta_c(t)dt = 1, \quad \epsilon > 0, \tag{16}$$

$$\int_{c-\epsilon}^{c+\epsilon} \delta_c(t)f(t)dt = f(c), \quad \epsilon > 0, \tag{17}$$

which represents a single instantaneous pulse, with one unit of area under its 'curve', taking place at some time c. If you find this definition troubling then you are not alone. No reasonable function can have this behavior and while it was invented by physicists to represent point sources of mass or charge, you can be reassured that mathematicians have made such a thing rigorous through the theory of distributions.[11]

3.1. **Mass-Spring Systems.** Again we investigate the forced mass spring system given by,

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = f(t), \quad m, b, k \in R^+ \cup \{0\}. \tag{18}$$

3.1.1. *Time Delayed Resonant Forcing.* Graph $f(t) = u_4(t)\cos(2(t-4))$ and solve (18) with $m = 1$, $b = 0$, $k = 2$ assuming that the system is set into motion with the mass at $y(0) = 1$ and with no initial velocity. Describe the long term behavior of the oscillator.

3.1.2. *Periodic Delta Forcing.* Assume the same initial conditions as above and using $f(t) = \sum_{n=1}^{\infty} \delta_n(t)$ find an expression for the position of the mass at any time, t. Describe the long term behavior of the oscillator.

3.1.3. *Single Delta Input.* Suppose we have that $b = 0$, $y(0) = \alpha$, $y'(0) = \beta$ and $f(t) = A\delta_T(t)$, show that the solution to (18) subject to these constraints is given by,

$$y(t) = \alpha\cos(\omega t) + \frac{\beta}{\omega}\sin(\omega t) + \frac{A}{m\omega}u_T(t)\sin(\omega(t-T)), \tag{19}$$

where $\omega^2 = \dfrac{k}{m}$.

3.1.4. *No breathe, no life.* Suppose that we wish to hit the mass in such a way that after the impact the oscillations stop. Show that for this to occur we must choose,

$$A = \frac{\alpha m\omega}{\sin(\omega T)} \tag{20}$$

$$T = \frac{1}{\omega}\arctan\left(-\frac{\alpha\omega}{\beta}\right). \tag{21}$$

3.2. **Shock Absorbers.** From Differential Equations: A Modeling Perspective, Borrelli, Coleman, 2004, we have the spotlight on modeling concerning shock absorption systems for automobiles. This mathematical model attempts to address the question, 'How do the strength and flexibility of shock absorbers affect the vibration of an automobile traveling on a bumpy road?'

3.2.1. *Reading Questions.* Read the attached material and respond to the following questions:

(1) What is the model equation?
(2) What do the parameters of this equation model?
(3) What are the assumptions on the forces involved?[12]
(4) What is the question the model equations seeks to address?
(5) What is the form of the steady-state solution?
(6) Compare the long-term behavior of the solution with initial conditions given by, $y(0) = 1$, $y'(0) = -1$, to the solution whose initial conditions are given by, $y(0) = 0$, $y'(0) = 2$.
(7) How does the amplitude of the stead-state response relate to the frequency of forcing?

[11] In the theory of distributions a generalized function is thought of not as a function itself, but only in relation to how it affects other functions when it is "integrated" against them.

[12] In 1994, I think, researchers finally solved analytically the problem of quadratic friction.

3.3. **Green's Functions.** Recall the differential equation given by (18). For this equation we can define the Green's function for the differential equation as the function g, which satisfies the analogous differential equation, [13]

$$m\frac{d^2g}{dt^2} + b\frac{dg}{dt} + kg = \delta_0(t), \quad g(0) = 0, \quad g'(0) = 0. \tag{22}$$

3.3.1. *Various Green's Functions.* Determine the Green's functions to (18) for:[14]

(1) $m = 1$, $b = -2$, $k = -3$
(2) $m = 1$, $b = 4$, $k = 4$
(3) $m = 1$, $b = -4$, $k = -13$
(4) $m = 1$, $b = 0$, $k = 4$

3.4. **Convolution.** Consider the following formula associated with products in the Laplace domain,

$$(f \star g) = \int_0^t f(t-u)g(u)du = \int_0^t f(u)g(t-u)du = \mathcal{L}^{-1}\left[F(s)G(s)\right], \tag{23}$$

where F, G are the Laplace transforms of f, g, respectively. The previous integral, (23), is called a convolution integral and gives us a general way to take products back from the Laplace domain to the time domain.[15] Using this inverse transformation it is possible to find a general solution to,

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = f(t), \quad y(0) = y_0, \quad y'(0) = y_0'. \tag{24}$$

3.4.1. *General Solution in Laplace Domain.* Solve (24) in the Laplace domain.
Take the Laplace transform of both sides of (24). Since the function $f(t)$ is unknown write it's Laplace transform as $F(s)$. Solving for $Y(s)$ in the Laplace domain gives,

$$Y(s) = (sy_0 + sy_0' + y_0)G(s) + F(s)G(s), \tag{25}$$

where $G(s) = (as^2 + bs + c)^{-1}$.[16]

3.4.2. *Solutions in Terms of Convolution Integrals.* Now assume that $y_0 = y_0' = 0$ so that the system is not initially perturbed and using convolution, (23), represent the solution, in the time domain, as an integral.

3.4.3. *Computation of Convolution Integral.* Assume that $m = 1$, $b = 0$, $k = 4$ and that $f(t) = e^{-t}$. Using convolutions compute the solution to the non-homogenous ODE.[17]

4. Power Series and Quantum Mechanics

5. Phase Analysis of Systems and the Trace Determinant Plane

6. Nonlinear Systems and Linearization

[13] A Green's function is a certain type of function used to solve nonhomogenous equations by considering the response of the system to a primitive external impulse, i.e. the Dirac-Delta *function*, and then using the primitive response to construct solutions for more complicated external forces. In physics Green's functions are often called propagators. In statistics a Green's function are often seen as correlation functions used to describe relationships between random variables.

[14] To do this take the Laplace transform of (22), solve for $G(s)$ and from $G(s)$ use tables to determine $g(t)$.

[15] Convolution integrals are powerful tools used to study differential equations. Many important statements can be made using convolution integrals. In statistics convolution is used to add probability density functions. In physics convolution integrals can be used to characterized heat evolution with decaying exponential functions, thus implying that it is a dissipative phenomenon. In optics one can represent the image stored on a camera as the physical image convolved with the geometry of the camera lens.

[16] Notice that the characteristic equation persist even using other techniques like integral transforms.

[17] There should be no need for re-derivations. The Green's function was already calculate in a previous step. The integral representing the solution in the time domain should be known!