MATH348-Advanced Engineering Mathematics		**Homework: LA Part II-Solutions**
Introduction to Vector Spaces, Eigenproblems and Diagonalization		
Text: 7.4, 7.9,	Lecture Notes: N/A	Slides: N/A

Quote of Homework: Linear Algebra Part II - Solutions	
Walk without rhythm, and it won't attract the worm. If you walk without rhythm, you never learn.	
	Norman Quentin Cook : Weapon of Choice (2000)

1. Vocabulary of Vector Spaces

Given,

$$\mathbf{A}_1 = \begin{bmatrix} 5 & 3 \\ -4 & 7 \\ 9 & -2 \end{bmatrix}, \quad \mathbf{b}_1 = \begin{bmatrix} 22 \\ 20 \\ 15 \end{bmatrix},$$

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -5 \\ 7 \\ 8 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ h \end{bmatrix},$$

$$\mathbf{w}_1 = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \quad \mathbf{w}_2 = \begin{bmatrix} -3 \\ 9 \\ -6 \end{bmatrix}, \quad \mathbf{w}_3 = \begin{bmatrix} 5 \\ -7 \\ h \end{bmatrix},$$

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix},$$

$$\mathbf{A}_2 = \begin{bmatrix} -8 & -2 & -9 \\ 6 & 4 & 8 \\ 4 & 0 & 4 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}.$$

Before we get into these problems we record the following row-reductions:

$$[\mathbf{A}_1 \,|\mathbf{b}_1] \sim \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \tag{1}$$

$$\mathbf{V} = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3] = \begin{bmatrix} 1 & -5 & 1 \\ -1 & 7 & 1 \\ -3 & 8 & h \end{bmatrix} \sim \left[\begin{array}{ccc|c} 1 & -5 & 1 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 2h+20 & 0 \end{array}\right] \tag{2}$$

$$\mathbf{W} = [\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3] = \begin{bmatrix} 1 & -3 & 5 \\ -3 & 9 & -7 \\ 2 & -6 & h \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 5 \\ 0 & 0 & 8 \\ 0 & 0 & h-10 \end{bmatrix} \tag{3}$$

$$[\mathbf{X}|\mathbf{y}] = [\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3|\mathbf{y}] = \left[\begin{array}{ccc|c} 1 & 2 & 4 & 3 \\ 0 & 1 & 2 & 1 \\ -1 & 3 & 6 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \tag{4}$$

$$[\mathbf{A}_2|\mathbf{b}_2] \sim \left[\begin{array}{ccc|c} -8 & -2 & -9 & 2 \\ 0 & 20 & 10 & 20 \\ 0 & 0 & 0 & 0 \end{array}\right] \tag{5}$$

1.1. **Linear Combinations.** Is $\mathbf{b}_1$ a linear combination of the columns of $\mathbf{A}_1$?

Recall the the following equivalence for $\mathbf{A} \in \mathbb{R}^{m\times n}$,

$$\mathbf{A}\mathbf{x} = \sum_{i=1}^{n} x_i \mathbf{a}_i, \tag{6}$$

which says that the matrix product is the same as a linear combination of its columns. So, asking if a vector, $\mathbf{b}$, is a linear combination of columns is also asking if $\mathbf{A}\mathbf{x} = \mathbf{b}$ has a solution. Either way we study the pivot structure of $[\mathbf{A}|\mathbf{b}]$. Thus, from above we have that $\mathbf{A}_1\mathbf{x} = \mathbf{b}_1$ has no solution and therefore $\mathbf{b}_1$ cannot be written as a linear combination of the columns from $\mathbf{A}$.

1.2. **Linear Dependence.** Determine all values for h such that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ forms a linearly dependent set.

A set of n-many vectors, $\mathbf{v}_i$, forms a linearly independent set if and only if $c_i = 0$ for $i = 1, 2, 3, \ldots, n$ is the only solution to $\sum_{i=1}^{n} c_i \mathbf{v}_i = \mathbf{0}$. This is equivalent to asking if $\mathbf{V}\mathbf{c} = \mathbf{0}$ has only the trivial solution, where $\mathbf{V}$ is a matrix formed by the set of vectors. If a homogeneous system has the trivial solution then there must be a pivot for every variable. If we want the vectors to form a linearly **de**pendent set then we must have the existence of free variables. Thus, from above, we require $h = -10$.

1.3. **Linear Independence.** Determine all values for h such that $S = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ forms a linearly independent set.

We repeat the argument of 1.2 and now require a pivot for each variable. In this case we have no values of h such that $c_1 = c_2 = c_3 = 0$ is the only solution to $\sum_{i=1}^{3} c_i \mathbf{w}_i = 0$. Thus, the vectors ALWAYS form a linearly dependent set.

1.4. **Spanning Sets.** How many vectors are in $S = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$? How many vectors are in span(S)? Is $\mathbf{y} \in$span(S)?

This is all a question of language. The set S has three elements. However, the spanning set of S is the set of all linear combinations of the vectors in S. That is, the spanning set of S is every vector that takes the form, $\sum_{i=1}^{3} c_i \mathbf{x}_i$ for any $c_i \in \mathbb{R}$. This spanning set, by definition has infinitely many elements.[1] Finally, we ask if $\mathbf{y}$ is in this spanning set, which is really asking if there are c_i's such that $\mathbf{y}$ can be written as $\sum_{i=1}^{3} c_i \mathbf{x}_i$. Again, this is the same as asking if $\mathbf{X}\mathbf{c} = \mathbf{y}$, which is addressed by understanding the solubility of $[\mathbf{X}|\mathbf{y}]$. From the previous row-reductions we see that this system has a solution, in fact is has many, and therefore $\mathbf{y} \in$ span(S).

1.5. **Matrix Spaces.** Is $\mathbf{b}_2 \in$ Nul$(\mathbf{A}_2)$? Is $\mathbf{b}_2 \in$ Col$(\mathbf{A}_2)$?

Recall that the null-space of a matrix is the set of all solutions to $\mathbf{A}\mathbf{x} = \mathbf{0}$. This space tells us about all the points in space the homogeneous linear equations simultaneously intersect. One way to determine if $\mathbf{b}_2$ is in the null-space of $\mathbf{A}_2$ is by solving the homogeneous equation and determining if $\mathbf{b}_2$ is one of these solutions. However, it pays to note that if $\mathbf{b}_2$ is in the null-space of $\mathbf{A}_2$ then $\mathbf{A}_2\mathbf{b}_2 = \mathbf{0}$. A quick check shows,

$$\mathbf{A}_2\mathbf{b}_2 = \begin{bmatrix} -8 & -2 & -9 \\ 6 & 4 & 8 \\ 4 & 0 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = \mathbf{0}. \tag{7}$$

The column space, on the other hand, is a little different. The column space is the set of all linear combinations of the columns of $\mathbf{A}_2$. This is also called the spanning set of the columns of $\mathbf{A}_2$. Thus, this question can be addressed in the same way as problem 1.1 or the last part of 1.4 and we have,

$$[\mathbf{A}_2|\mathbf{b}_2] \sim \left[\begin{array}{ccc|c} -8 & -2 & -9 & 2 \\ 0 & 20 & 10 & 20 \\ 0 & 0 & 0 & 0 \end{array}\right]. \tag{8}$$

The conclusion is that $\mathbf{b}_2$ is in both the null-space and column space of $\mathbf{A}_2$. This is not generally true of a non-trivial vector. In fact, it is never true for rectangular coefficient data.

[1] As a simple case consider every scaling of $\hat{\mathbf{i}}$ how many elements would be in this set?

2. The Spaces Defined by Linear Transformations

Given,

$$\mathbf{A} = \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ -2 & 3 & -3 & -3 & -4 \\ 4 & -6 & 9 & 5 & 9 \\ -2 & 3 & 3 & -4 & 1 \end{bmatrix}.$$

The following problems will require the use of an echelon form of $\mathbf{A}$. One such form is,

$$\mathbf{A} \sim \begin{bmatrix} 2 & -3 & 6 & 2 & 5 \\ 0 & 0 & 3 & -1 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \mathbf{B}. \tag{9}$$

2.1. **Null Space.** Determine a basis and the dimension of Nul($\mathbf{A}$).

The null-space of $\mathbf{A}$ is the set of all solutions to $\mathbf{Ax} = \mathbf{0}$. To find a basis for this space we must explicitly solve the homogeneous equation. Thus, from the echelon form $\mathbf{B}$ we have the following,

$$\begin{aligned} x_4 &= -3x_5 \\ x_3 &= (x_4 - x_5)/3 = (-3x_5 - x_5)/3 = -\frac{4}{3}x_5 \\ x_1 &= \frac{1}{2}(3x_2 - 6x_3 - 2x_4 - 5x_5) = \frac{1}{2}(3x_2 - 6(-\frac{4}{3}x_5) - 2(-3x_5) - 5x_5) = \\ &= \frac{1}{2}(3x_2 + 8x_5 + 6x_5 - 5x_5) = \frac{3}{2}x_2 + \frac{9}{2}x_5 \\ x_2 &\in \mathbb{R} \\ x_5 &\in \mathbb{R} \\ &\Rightarrow \quad \mathbf{x} = x_2 \begin{bmatrix} 3/2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -9/2 \\ 0 \\ -4/3 \\ -3 \\ 1 \end{bmatrix} \qquad x_2, x_5 \in \mathbb{R} \end{aligned}$$

Hence, the basis for Nul($\mathbf{A}$) is

$$B_{null} = \left\{ \begin{bmatrix} 3/2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -9/2 \\ 0 \\ -4/3 \\ -3 \\ 1 \end{bmatrix} \right\} \tag{10}$$

and dim(Nul $\mathbf{A}$) = 2. The conclusion is that the five linear four-dimensional objects intersect at many points in $\mathbb{R}^5$. The collection of points forms a two-dimensional subspace, which is spanned by the basis vectors. That is, the linear objects intersect forming a planer subspace of $\mathbb{R}^5$. [2]

2.2. **Column Space.** Determine a basis and the dimension of Col($\mathbf{A}$).
The column-space is the set of all linear combinations of the columns of $\mathbf{A}$. We would like to know a basis for this space, which implies that we must somehow determine the columns of the $\mathbf{A}$ matrix that contain unique directional information. That is, we must find the linearly independent columns of the $\mathbf{A}$ matrix. This information has been made clear through the previous null-space problem. Recall that if a set of vectors is linearly independent then their corresponding homogeneous equation must only have the trivial solution. Since row-reduction does not change the solution to a homogeneous equation we have,

$$\mathbf{Ax} = \mathbf{0} \iff [\mathbf{A}|\mathbf{0}] \sim [\mathbf{B}|\mathbf{0}] \iff \mathbf{Bx} = \mathbf{0}. \tag{11}$$

So, we can see if the columns of $\mathbf{A}$ are linearly independent by considering the linear independence of the columns of $\mathbf{B}$. Clearly, the previous problem shows that the columns of $\mathbf{B}$ are not linearly independent. However, it is also clear from $\mathbf{B}$ that the columns without pivots can be made using the columns with pivots, $\mathbf{b}_2 = -3/2\mathbf{b}_1$ and $\mathbf{b}_5 = 3\mathbf{b}_4 + 4/3\mathbf{b}_3 - (9/2)\mathbf{b}_1$. So, if we take only the pivot columns

[2]It is important to notice how the dimension of the null-space drives the previous statements. I did not draw or try to picture anything.

from $\mathbf{B}$ then we would loose the linearly dependent columns and their free-variables. Consequently, the only solution to $\mathbf{B}_{change} = \mathbf{0}$ would be the trivial solution, which implies the columns are linearly independent.

There is still a problem. While row-reduction did not change the dependence relation, it did change the actual vectors. That is, the column-space of $\mathbf{A}$ is different the the column-space of $\mathbf{B}$. To see this consider the constants neceesary for $\mathbf{a}_1 \stackrel{?}{=} c_1\mathbf{b}_1 + c_2\mathbf{b}_3 + c_3\mathbf{b}_4$. [3] So, the conclusion is that we must take the linearly independent columns from $\mathbf{A}$ as told to us by $\mathbf{B}$. Thus, a basis for the column space of $\mathbf{A}$ are the pivot columns of $\mathbf{A}$,

$$B_{ColA} = \left\{ \begin{bmatrix} 2 \\ -2 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} 6 \\ -3 \\ 9 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 5 \\ -4 \end{bmatrix} \right\} \tag{12}$$

and $\dim(\text{Col}\mathbf{A}) = 3$. The dimension of the column-space is also known as the rank of $\mathbf{A}$. From this we see an example of the so-called rank-nullity theorem, which says that the dimension of the null-space and the dimension of the column-space must always add to be the total number of columns in the matrix. That is,

$$\text{Rank } \mathbf{A} + \dim(\text{Nul}(\mathbf{A})) = n, \text{ where } \mathbf{A} \in \mathbb{R}^{m\times n}.$$

2.3. **Row Space.** Determine a basis and the dimension of Row $\mathbf{A}$. What is the Rank of $\mathbf{A}$?

The row-space of a matrix is the set of all linear combinations of its rows. A basis can be found by taking only the linearly independent rows of the matrix, which can be clearly seen as the non-zero rows of any echelon form. While in the case of a column-space the columns must necessarily come from the original matrix, this is not a requirement for the row-space.[4] Thus, a basis for the row-space of $\mathbf{A}$ is given by,

$$B_{RowA} = \left\{ \begin{matrix} [2\ -3\,6\,2\,5] \\ [0\,0\,3\ -1\,1] \\ [0\,0\,0\,1\,3] \end{matrix} \right\}. \tag{13}$$

Since these rows were chosen because of their pivots, the dimension of this space is always equal to the dimension of the column space and $\dim(\text{Row}\mathbf{A}) = \text{Rank } \mathbf{A} = 3$. [5]

3. Eigenvalues and Eigenvectors

Given,

$$\mathbf{A}_1 = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}, \quad \mathbf{A}_3 = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}, \quad \mathbf{A}_4 = \begin{bmatrix} .1 & .6 \\ .9 & .4 \end{bmatrix}, \quad \mathbf{A}_5 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix},$$

3.1. **Eigenproblems.** Find all eigenvalues and eigenvectors of $\mathbf{A}_i$ for $i = 1, 2, 3, 4, 5$.

Recall that the associated eigenproblem for a square matrix $\mathbf{A}_{n\times n}$ is defined by $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$ whose solution is found via the following auxiliary equations:

- Characteristic Polynomial : $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$
- Associated Null-space : $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$

For each of the previous matrices we have:

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[3] Answer : There are no constants that allow for this to be true.

[4] The reason for this is that row-operations <u>are</u> linear combinations, $R_i = R_i + \alpha R_j$. Thus using the non-zero rows of any echelon form you can get back to the rows of the original matrix and all linear combinations for that matter.

[5] It is possible to take the corresponding rows from $\mathbf{A}$ but dangerous. The reason why is that the rows of the echelon form may not correspond directly to the rows of the original matrix because of row-swaps. However, if you wanted to take the rows from $\mathbf{A}$ and have kept track of your row-swaps then there shouldn't be a problem.

$$
\begin{aligned}
\det(\mathbf{A}_1 - \lambda\mathbf{I}) &= (4-\lambda)(1-\lambda)^2 + 2(1-\lambda) \\
&= (1-\lambda)(\lambda^2 - 5\lambda + 6) = 0 \implies \lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3
\end{aligned}
$$

<u>Case $\lambda_1 = 1$:</u>

$$
\left[\begin{array}{c|c} \mathbf{A}_1 - \lambda_1\mathbf{I} & \mathbf{0} \end{array}\right] = \left[\begin{array}{ccc|c} 3 & 0 & 1 & 0 \\ -2 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 3 & 0 & 1 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} 3x_1 = -x_3 \\ -2x_1 = 0 \\ x_2 \in \mathbb{R} \end{array} \Rightarrow \mathbf{x} = \begin{bmatrix} 0 \\ x_2 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}
$$

A basis for this eigenspace associated with $\lambda = 1$ is $B_{\lambda=1} = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

<u>Case $\lambda_2 = 2$:</u>

$$
\begin{aligned}
\left[\begin{array}{c|c} \mathbf{A}_1 - \lambda_2\mathbf{I} & \mathbf{0} \end{array}\right] &= \left[\begin{array}{ccc|c} 2 & 0 & 1 & 0 \\ -2 & -1 & 0 & 0 \\ -2 & 0 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & 0 & 1 & 0 \\ -2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \\
&\sim \left[\begin{array}{ccc|c} 2 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_1 = -x_3/2 \\ x_2 = x_3 \\ x_3 \in \mathbb{R} \end{array} \Rightarrow \mathbf{x} = \begin{bmatrix} -1/2 \\ 1 \\ 1 \end{bmatrix} x_3
\end{aligned}
$$

A basis for this eigenspace is $B_{\lambda=2} = \left\{ \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} \right\}$

<u>Case $\lambda_3 = 3$:</u>

$$
\begin{aligned}
\left[\begin{array}{c|c} \mathbf{A}_1 - \lambda_3\mathbf{I} & \mathbf{0} \end{array}\right] &= \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ -2 & -2 & 0 & 0 \\ -2 & 0 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \\
&\Rightarrow \begin{array}{l} x_1 = -x_3 \\ x_2 = x_3 \\ x_3 \in \mathbb{R} \end{array} \Rightarrow \mathbf{x} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} x_3
\end{aligned}
$$

A basis for this eigenspace is $B_{\lambda=3} = \left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

$$
\det(\mathbf{A}_2 - \lambda\mathbf{I}) = (3-\lambda)(1-\lambda) - (-2) = \lambda^2 - 4\lambda + 5 \implies \lambda = \frac{-(-4) \pm \sqrt{16 - 4(1)(5)}}{2} = 2 \pm i
$$

<u>Case $\lambda = 2 \pm i$:</u>

$$
\begin{aligned}
\left[\begin{array}{c|c} \mathbf{A}_2 - \lambda\mathbf{I} & \mathbf{0} \end{array}\right] &= \left[\begin{array}{cc|c} 3-(2\pm i) & 1 & 0 \\ -2 & 1-(2\pm i) & 0 \end{array}\right] = \\
&= \left[\begin{array}{cc|c} 1 \mp i & 1 & 0 \\ -2 & -1 \mp i & 0 \end{array}\right].
\end{aligned}
$$

Row-reduction with complex numbers is possible. However, it is easier to note that for a two-by-two system we can use either row, in this case the first,

$$
(1 \mp i)x_1 + 1x_2 = 0 \iff (1 \mp i)x_1 = -x_2, \tag{14}
$$

to define the ratio between x_1 and x_2.[6] That is, if $x_1 = -1$ then $x_2 = 1 \mp i$ and thus the eigenvectors, like the eigenvalues, come in complex conjugate pairs $\mathbf{x} = [-1 \;\; 1 \mp i]^{\mathrm{T}}$.[7]

[6]This only works for two-by-two problems. In higher dimensions it is not possible to fix one variable and uniquely define the remaining variables.

[7]For real matrices, complex Eigenvalues and eigenvectors must occur in conjugate pairs.

Since $\mathbf{A}_3$ is triangular we know the eigenvalues of$\mathbf{A}_3$ are,

$$\begin{aligned} \lambda_1 &= 4 \qquad \text{(With algebraic multiplicity of 2)}, \\ \lambda_2 &= 2 \qquad \text{(With algebraic multiplicity of 2)}. \end{aligned}$$

Case $\lambda_1 = 4$:

$$\left[\begin{array}{c|c} \mathbf{A}_3 - \lambda_1 \mathbf{I} & \mathbf{0} \end{array}\right] = \left[\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 1 & 0 & 0 & -2 & 0 \end{array}\right] \Rightarrow \begin{array}{c} -2x_3 = 0 \\ x_1 = 2x_4 \\ x_2, x_4 \in \mathbb{R} \end{array} \Rightarrow \mathbf{x} = \begin{bmatrix} 2x_4 \\ x_2 \\ 0 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Thus the basis for this eigenspace is $B_{\lambda=4} = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Case $\lambda = 2$:

$$\left[\begin{array}{c|c} \mathbf{A}_3 - \lambda_2 \mathbf{I} & \mathbf{0} \end{array}\right] = \left[\begin{array}{cccc|c} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{c} x_1 = 0 \\ x_2 = 0 \\ x_3, x_4 \in \mathbb{R} \end{array} \Rightarrow \mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

A basis for this eigenspace is $B_{\lambda=2} = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

$$\begin{aligned} \det\left(\begin{bmatrix} .1 - \lambda & .6 \\ .9 & .4 - \lambda \end{bmatrix}\right) &= (.4 - \lambda)(.1 - \lambda) - .54 = \lambda^2 - .5\lambda - .54 + .04 = \\ &= \lambda^2 - .5\lambda - .5 \Rightarrow \lambda = \frac{-(-.5) \pm \sqrt{(-.5)^2 - 4(1)(-.5)}}{2(1)} = \frac{.5 \pm 1.5}{2} \Rightarrow \lambda_1 = 1,\ \lambda_2 = -.5 \end{aligned}$$

Case $\lambda_1 = 1$:

(15)
$$[\mathbf{A}_4 - \lambda_1 \mathbf{I} | \mathbf{0}] = \left[\begin{array}{cc|c} -.9 & .6 & 0 \\ .9 & .6 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} -.9 & .6 & 0 \\ 0 & 0 & 0 \end{array}\right] \Longrightarrow \mathbf{x}_1 = \begin{bmatrix} 2/5 \\ 3/5 \end{bmatrix}$$

Case $\lambda_2 = -.5$:

(16)
$$[\mathbf{A}_4 - \lambda_2 \mathbf{I} | \mathbf{0}] = \left[\begin{array}{cc|c} .6 & .6 & 0 \\ .9 & .9 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} .6 & .6 & 0 \\ 0 & 0 & 0 \end{array}\right] \Longrightarrow \mathbf{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

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(17)
$$\det(\mathbf{A}_5 - \lambda \mathbf{I}) = \lambda^2 - 1 = 0 \Longrightarrow \lambda = \pm 1$$

Case $\lambda = \pm 1$:

(18)
$$[\mathbf{A}_5 - \lambda \mathbf{I} | \mathbf{0}] = \left[\begin{array}{cc|c} \mp 1 & -i & 0 \\ i & \mp 1 & 0 \end{array}\right] \Longrightarrow \mp x_1 - i x_2 = 0 \Longleftrightarrow \mp x_1 = i x_2 \Longrightarrow \mathbf{x} = \begin{bmatrix} i \\ \mp 1 \end{bmatrix}$$

3.2. **Diagonalization.** Find all matrices associated with the diagonalization of $\mathbf{A}_i$ for $i = 1, 2, 3, 4, 5$.

For each of the matrices we must find $\mathbf{P}$,$\mathbf{D}$ and $\mathbf{P}^{-1}$ such that $\mathbf{A}_i = \mathbf{PDP}^{-1}$ for $i = 3, 4, 5$. Recall:
If one finds $n-$many eigenvectors for an $n \times n$ matrix then it is possible to find a diagonal matrix *similar* to $\mathbf{A}_{n\times n}$. That is, if $\mathbf{A}$ has $n-$many eigenvectors then $\mathbf{A}$ has the following diagonal decomposition,

$$\mathbf{A} = \mathbf{PDP}^{-1}, \tag{19}$$

where $\mathbf{D}$ is a diagonal matrix whose elements are eigenvalues of $\mathbf{A}$ and $\mathbf{P}$ is an invertible matrix whose columns are the eigenvectors corresponding to the eigenvalue elements of $\mathbf{D}$. For $i = 3, 4, 5$ we have the following.

$$[\mathbf{P}_3|\mathbf{I}] \sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1/2 & 0 & 0 & 1 \end{array}\right] = \left[\mathbf{I}|\mathbf{P}_3^{-1}\right], \text{ and } \mathbf{D}_3 = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \tag{20}$$

$$\mathbf{P}_4 = \begin{bmatrix} 2/5 & 1 \\ 3/5 & -1 \end{bmatrix} \implies \mathbf{P}_4^{-1} = \begin{bmatrix} 1 & 1 \\ 3/5 & -2/5 \end{bmatrix}, \text{ and } \mathbf{D}_4 = \begin{bmatrix} 1 & 0 \\ 0 & -1/2 \end{bmatrix} \tag{21}$$

$$\mathbf{P}_5 = \begin{bmatrix} i & i \\ -1 & 1 \end{bmatrix} \implies \mathbf{P}_5^{-1} = \frac{1}{2i}\begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} = \frac{i}{2}\begin{bmatrix} -1 & i \\ -1 & -i \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -i & -1 \\ -i & 1 \end{bmatrix}, \text{ and } \mathbf{D}_5 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{22}$$

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4. Regular Stochastic Matrices

For the *regular stochastic matrix* $\mathbf{A}_4$, define its associated steady-state vector, $\mathbf{q}$, to be such that $\mathbf{A}_4\mathbf{q} = \mathbf{q}$.

4.1. **Limits of Time Series.** Show that $\lim_{n\to\infty} \mathbf{A}_4^n\mathbf{x} = \mathbf{q}$ where $\mathbf{x} \in \mathbb{R}^2$ such that $x_1 + x_2 = 1$.

First, we note that we have already found the steady-state vector $\mathbf{q}$ since it is the eigenvector associated with $\lambda_1 = 1$. Now, the question is how to raise a matrix to an infinite power. Generally, it is unclear whether this processes converges and if it does, what it converges to. However, diagonalization offers us hope since,

$$\lim_{n\to\infty} \mathbf{A}^n = \lim_{n\to\infty} \mathbf{PD}^n\mathbf{P}^{-1} = \mathbf{P} \lim_{n\to\infty} \mathbf{D}^n\mathbf{P}^{-1}, \tag{23}$$

where $[\mathbf{D}^n]_{ij} = d_{ii}^2\delta_{ij}$. Though calculating $\mathbf{A}^n$ is hard, calculating $\mathbf{D}^n$ is easy and more importantly, limiting processes on matrices now reduce to limiting processes on scalars, which is well-understood. In this case we have,

$$\lim_{n\to\infty} \mathbf{A}_4^n\mathbf{x} = \lim_{n\to\infty} \mathbf{P}_4\mathbf{D}_4^n\mathbf{P}_4^{-1}\mathbf{x} = \begin{bmatrix} 2/5 & 1 \\ 3/5 & -1 \end{bmatrix}\begin{bmatrix} \lim_{n\to\infty} 1^n & 0 \\ 0 & \lim_{n\to\infty} (-.5)^n \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 3/5 & -2/5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{24}$$

$$= \begin{bmatrix} .4 & .4 \\ .6 & .6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} .4(x_1 + x_2) \\ .6(x_1 + x_2) \end{bmatrix} = \begin{bmatrix} .4 \\ .6 \end{bmatrix} = \begin{bmatrix} 2/5 \\ 3/5 \end{bmatrix} = \mathbf{q}. \tag{25}$$

5. Orthogonal Diagonalization and Spectral Decomposition

Recall that if $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$ then their inner-product is defined to be $\langle \mathbf{x}, \mathbf{y}\rangle = \mathbf{x}^{\mathrm{H}}\mathbf{y} = \bar{\mathbf{x}}^{\mathrm{T}}\mathbf{y}$. Also, in this case, the 'length' of the vector is taken to be $|\mathbf{x}| = \sqrt{\langle \mathbf{x}, \mathbf{x}\rangle}$.

5.1. **Self-Adjointness.** Show that $\mathbf{A}_5$ is a self-adjoint matrix.

First note that $\mathbf{A}_5 = \sigma_y$ from homework one. It was shown in this homework that $\mathbf{A}_5^{\mathrm{H}} = \mathbf{A}_5$.

5.2. **Orthogonal Eigenvectors.** Show that the eigenvectors of $\mathbf{A}_5$ are orthogonal with respect to the inner-product defined above.

Vectors are orthogonal if their inner-product is zero. With our previous definition of inner-product, the calculation,

$$\mathbf{x}_{\mp}^{\mathrm{H}}\mathbf{x}_{\pm} = [\bar{i} \ \mp 1] \begin{bmatrix} i \\ \mp 1 \end{bmatrix} = [-i \ \pm 1] \begin{bmatrix} i \\ \mp 1 \end{bmatrix} = -i \cdot i + \mp 1 \cdot \pm 1 = 1 - 1 = 0, \tag{26}$$

shows that the eigenvectors are orthogonal.

5.3. **Orthonormal Eigenbasis.** Using the previous definition for length of a vector and the eigenvectors of the self-adjoint matrix, construct an *orthonormal basis* for $\mathbb{C}^2$.

An orthonormal basis is an orthogonal basis where the basis vectors have all been scaled to have unit-length. Using our definition of inner-product to define a length we note,

$$\sqrt{\mathbf{x}_{\mp}^{\mathrm{H}}\mathbf{x}_{\mp}} = \sqrt{1+1} = \sqrt{2}, \tag{27}$$

which implies that the normalized eigenvectors are, $\mathbf{x}_{\mp} = [i\sqrt{2}/2 \quad \mp\sqrt{2}/2]^{\mathrm{T}}$.

5.4. **Orthogonal Diagonalization.** Show that $\mathbf{U}^{\mathrm{H}} = \mathbf{U}^{-1}$, where $\mathbf{U}$ is a matrix containing the normalized eigenvectors of $\mathbf{A}_5$.

We have seen from the previous problems that if you have enough eigenvectors then it is possible to find a diagonal decomposition for the matrix. Geometrically, this decomposition provides a natural coordinate system for which the solution to the associated linear problem is manifestly clear. This is a powerful result but it can be made stronger.

The general statement is,

- If a matrix is self-adjoint then it can always be diagonalized.[8] Moreover, eigenvectors associated with different eigenvalues are orthogonal to one another and the resulting matrix can be constructed to have the property $\mathbf{P}\mathbf{P}^{\mathrm{H}} = \mathbf{P}^{\mathrm{H}}\mathbf{P} = \mathbf{I}$.[9]

Since $\mathbf{A}_5$ is self-adjoint we an demonstrate this fact.

$$\mathbf{U} = \begin{bmatrix} i\frac{\sqrt{2}}{2} & i\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \implies \mathbf{U}^{\mathrm{H}} = \begin{bmatrix} -i\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ -i\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \text{ and } \mathbf{U}\mathbf{U}^{\mathrm{H}} = \begin{bmatrix} \frac{1}{2}+\frac{1}{2} & 0 \\ 0 & \frac{1}{2}+\frac{1}{2} \end{bmatrix}. \tag{28}$$

Thus, $\mathbf{A}_5 = \mathbf{U}\mathbf{D}_5\mathbf{U}^{\mathrm{H}}$ where the decomposition has been found without using row-reduction to find an inverse matrix!

5.5. **Spectral Decomposition.** Show that $\mathbf{A}_5 = \lambda_1\mathbf{x}_1\mathbf{x}_1^{\mathrm{H}} + \lambda_2\mathbf{x}_2\mathbf{x}_2^{\mathrm{H}}$.

The previous result is quite powerful and can be used to derive other decompositions of the matrix $\mathbf{A}_5$. One such decomposition is called the spectral decomposition, which speaks to the action of $\mathbf{A}_5$ as a transformation. Assuming the given decomposition we consider the transformation,

$$\begin{aligned} \mathbf{A}_5\mathbf{y} &= \left(\lambda_1\mathbf{x}_1\mathbf{x}_1^{\mathrm{H}} + \lambda_2\mathbf{x}_2\mathbf{x}_2^{\mathrm{H}}\right)\mathbf{y} && (29) \\ &= \lambda_1\mathbf{x}_1\mathbf{x}_1^{\mathrm{H}}\mathbf{y} + \lambda_2\mathbf{x}_2\mathbf{x}_2^{\mathrm{H}}\mathbf{y} && (30) \\ &= \lambda_1\left\langle\mathbf{x}_1,\mathbf{y}\right\rangle\mathbf{x}_1 + \lambda_2\left\langle\mathbf{x}_2,\mathbf{y}\right\rangle\mathbf{x}_2 && (31) \end{aligned}$$

[8]It can also be shown that its eigenvalues are always real. This is important to the theory of quantum mechanics where the eigenvalues are hypothetical measurements associated with a quantum system. It would be disconcerting if you stuck a thermometer into a quantum-turkey and it somehow read $3+2i$. Yikes!

[9]If the eigenvectors from a shared eigenspace are not orthogonal then it is possible to orthogonalize them by the *Gram-Schmidt* process.

which implies that $\mathbf{A}_5$ transforms the vector $\mathbf{y}$ by projecting this vector onto each eigenvector, rescaling it by a factor of λ_i and then linearly combines the results. To demonstrate this decomposition we calculate the following outer-product,

$$\mathbf{x}_{\mp}\mathbf{x}_{\mp}^{\mathrm{H}} = \begin{bmatrix} i\frac{\sqrt{2}}{2} \\ \mp\frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} \bar{i}\sqrt{2}/2 & \mp\sqrt{2}/2 \end{bmatrix} \tag{32}$$

$$= \begin{bmatrix} i\frac{\sqrt{2}}{2} \\ \mp\frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} -i\sqrt{2}/2 & \mp\sqrt{2}/2 \end{bmatrix} \tag{33}$$

$$= \begin{bmatrix} 1/2 & \mp i/2 \\ \pm i/2 & 1/2 \end{bmatrix}, \tag{34}$$

which gives,

$$\mathbf{A}_5 = 1 \cdot \begin{bmatrix} 1/2 & -i/2 \\ i/2 & 1/2 \end{bmatrix} - 1 \cdot \begin{bmatrix} 1/2 & i/2 \\ -i/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}. \tag{35}$$