MATH348 - April 15, 2011 **Exam II - 50 Points**	**NAME:**

In order to receive full credit, SHOW ALL YOUR WORK. Full credit will be given only if all reasoning and work is provided. When applicable, please enclose your final answers in boxes.

1. (10 Points) Conceptual Questions. For the following questions, assume that we are considering the physical problem on a bounded domain, $x \in (0, 1)$.

   (a) Write down the heat **and** wave equations and any initial conditions needed for unique solutions.

   $$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

   $u(x,0) = f(x)$

   $u(x,0) = f(x)$
   $u_t(x,0) = g(x)$

   (b) Suppose we are given the boundary conditions $u(0, t) = 0$ and $u_x(1, t) = 0$ for each problem. Explain the physical meaning of **each** boundary condition for **both** the heat equation and wave equation.

   Heat: $0°$ temp on left, ideal insulation on Right

   wave: fixed on left, free on right

   (c) How do solutions of these heat and wave equations behave/evolve in time?

   Soln to heat Eqn decay in time while Soln to wave oscillate.

   (d) If $u(x, t)$ is an equilibrium solution, $\dfrac{\partial u}{\partial t} = 0$, for all t, to the heat equation then is it a solution to the wave equation? Explain.

   Yes. See (a) if $u_t = 0 \Rightarrow u_{tt} = 0$ and both PDE are the same.

2. (10 Points) Quickies

(a) Given,

$$F''(x) + \lambda F(x) = 0, \quad \lambda \in [0, \infty).$$

The following table contains different boundary conditions for the ODE. Fill in each table element with either a yes or a no.

	Boundary value problem has a cosine solution	Boundary value problem has a sine solution	Boundary value problem has a nontrivial constant solution
$F'(0) = 0,\ F'(L) = 0$	✓	✗	✓
$F(0) = 0,\ F'(L) = 0$	✗	✓	✗
$F(0) = 0,\ F(L) = 0$	✗	✓	✗
$F'(0) = 0,\ F(L) = 0$	✓	✗	✗

(b) Suppose that we know that,

$$\begin{aligned} G_n(t) &= B_n e^{-k_n c^2 t}. \quad B_n \in \mathbb{R}, \\ F_n(x) &= \cos(k_n x), \quad k_n = n\pi, \quad n = 0, 1, 2, \cdots, \end{aligned}$$

are the temporal and spatial solutions to some heat equation. Assuming that $u(x, 0) = f(x)$:

i. Write down the general solution to the PDE.

$$u(x,t) = \sum_{n=0}^{\infty} B_n e^{-k_n c^2 t} \cos(k_n x)$$

ii. Solve for any unknown constants in terms of $f(x)$.

$$B_0 = \frac{1}{L}\int_0^L f(x)\,dx, \quad B_n = \frac{2}{L}\int_0^L f(x)\cos(k_n x)\,dx$$

iii. What is long term behavior of the temperature of this one-dimensional object?

$$\lim_{t \to \infty} u(x,t) = B_0 = f_{avg}$$

3. (10 Points) Show that the function solves its associated differential equation.

(a) $u(x,y) = \ln\left(x^2+y^2\right)$ for $u_{xx} + u_{yy} = 0$.

$$\frac{\partial}{\partial x}\ln(x^2+y^2) = \frac{1}{x^2+y^2}\cdot 2x \Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{2}{x^2+y^2} - \frac{2x}{(x^2+y^2)^2}\cdot 2x$$

$$\Rightarrow u_{xx} + u_{yy} = \frac{4}{x^2+y^2} - \frac{4(x^2+y^2)}{x^2+y^2} = 0 \checkmark$$

(b) $u(x,t) = f(x-ct)$ for $u_{tt} = c^2 u_{xx}$.

$$\frac{\partial u}{\partial t} = \frac{\partial f(x-ct)}{\partial t} = f'\cdot(-c), \Rightarrow \frac{\partial^2 u}{\partial t^2} = c^2 f''$$

$$\frac{\partial^2 u}{\partial x^2} = f'' \Rightarrow u_{tt} = c^2 f'' = u_{xx} \checkmark$$

4. (10 Points) Using separation of variables define three ODEs consistent with the PDE,

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}. \quad (1)$$

$$u(x,y,z) = F(x,y)G(z)$$

$$\Rightarrow (1) \Leftrightarrow F_{xy}G = F_xG + F_{yy}G + FG''$$

$$\Rightarrow \frac{F_{xy} - F_x - F_{yy}}{F} = \frac{G''}{G} = -\lambda$$

$$F(x,y) = X(x)Y(y)$$

$$\Rightarrow F_{xy} - F_x - F_{yy} + \lambda F = X'Y' - X'Y - Y''X + \lambda XY = 0$$

$$\Rightarrow \frac{X'}{X}\left(\frac{Y'}{Y} - 1\right) - \frac{Y''}{Y} + \lambda = 0$$

$$\frac{X'}{X} = \frac{\frac{Y''}{Y} - \lambda}{\frac{Y'}{Y} - 1} = -\alpha \Rightarrow X' = -\alpha X, \quad Y'' - \lambda Y + \alpha(Y' - Y) = 0$$

5. (10 Points) Solve the following partial differential equation,

$$\frac{\partial^2 u}{\partial t^2} + 2\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad x \in \left(0, \frac{1}{2}\right), \quad t \in (0, \infty),$$

$$u(0,t) = 0, \quad u\left(\frac{1}{2}, t\right) = 0, \quad u(x,0) = 0, \quad u_t(x,0) = g(x).$$

See Fall 2011

Problem 5