MATH348 - August 11, 2010
Exam II - 50 Points

NAME: Key

In order to receive full credit, SHOW ALL YOUR WORK. Full credit will be given only if all reasoning and work is provided. When applicable, please enclose your final answers in boxes.

1. (10 Points)

(a) What types of functions can be represented by a Fourier series?

Periodic fn + Fn defined on finite domains of $\mathbb{R}$ s.t. $\int_D |f| < \infty$, where D is the principle domain.

(b) What types of functions can be represented by a Fourier integral?

Any fn defined on $\mathbb{R}$ s.t. $\int_{\mathbb{R}} |f| < \infty$. The fn need ~~in~~ not be periodic but through the use of delta fn it can be.

(c) What is the connection between Fourier series and Fourier integral?

If the principle domain of a Fourier ~~integral~~ series is [-L, L] then on can "derive" the Fourier by taking $\lim_{L\to\infty}$ F.S.(x)

2. (10 Points) Given the following integrals,

$$\int_{-\pi}^{\pi} f(x)dx = 5,$$

$$\int_{-\pi}^{\pi} f(x)\cos(nx)dx = \left[\frac{\cos(2nx)}{n}\Big|_{-\pi}^{0} + \frac{\cos(2nx)}{n}\Big|_{0}^{\pi} \right], = \frac{1}{n}\left[\cos(0) - \cos(-2\pi n) + \cos(2n\pi) - \cos(0)\right] = 0$$

$$\frac{1}{n}\left[1 - (-1)^n - (-1)^n + 1\right] = \int_{-\pi}^{\pi} f(x)\sin(nx)dx = \left[\frac{\cos(nx)}{n}\Big|_{-\pi}^{0} - \frac{\cos(nx)}{n}\Big|_{0}^{\pi} \right],$$

$$\int_{-\pi}^{\pi} g(x)dx = e^{-in\pi} - e^{in\pi} = (-1)^n - (-1)^n = 0$$

$$\int_{-\pi}^{\pi} g(x)e^{-inx}dx = \frac{i}{n}\left[e^{-inx}\Big|_{0}^{\pi} - e^{inx}\Big|_{-\pi}^{0} \right] = \frac{i}{n}\left[(-1)^n - 1 - \cancel{1} - (-1)^n\right]$$

$$\frac{2\cos(\omega)}{2} = \frac{e^{i\omega} + e^{-i\omega}}{2} = \int_{-\infty}^{\infty} h(x)e^{-i\omega x}dx,$$

where n is an integer and $\omega \in \mathbb{R}$.

(a) Calculate the real Fourier series of $f(x)$.

$$f(x) = \frac{5}{2\pi} + \sum_{n=1}^{\infty} \frac{2}{n\pi}\left[1 - (-1)^n\right]\sin(nx)$$

(b) Calculate the complex Fourier series of $g(x)$.

$$g(x) = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{2i}{2n\pi}\left[(-1)^n - 1\right]e^{inx}$$

(c) Calculate the Fourier transform of $h(x)$.

$$\hat{h}(\omega) = \frac{\cos(\omega)}{\sqrt{2\pi}}$$

(d) Determine the symmetry of the function $f(x)$.

Neither Even nor odd

(e) Calculate the real Fourier series representation of $g(x)$.

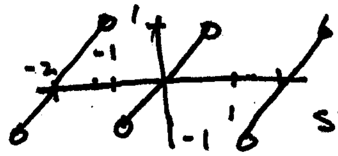
$$g(x) = \sum_{n=1}^{\infty} \frac{\cancel{2}i}{\cancel{2}n\pi}\left[(-1)^n - 1\right]e^{inx} - \sum_{n=1}^{\infty} \frac{i}{n\pi}\left[(-1)^n - 1\right]e^{-inx} =$$

$$= \sum_{n=1}^{\infty} 2 \cdot \frac{1-(-1)^n}{n\pi}\sin(nx)$$

3. (10 Points) Given that $f(x) = x$ for $x \in (0, 1)$

(a) On two separate graphs, sketch the Fourier cosine **and** Fourier sine half-range expansions of f.

Cosine



sine

(b) Calculate the Fourier cosine **and** Fourier sine half-range expansions of f.

Cosine: $a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{1}\int_0^1 f(x)\,dx = \frac{1}{2}$

$a_n = \frac{2}{1}\int_0^1 x\cos(n\pi x)\,dx = 2\left(\frac{\cos(n\pi)}{n^2\pi^2} - \frac{1}{n^2\pi^2}\right)$

$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2}\left((-1)^n - 1\right)\cos(n\pi x)$

u	dv
x	$\cos(n\pi x)$
1	$\sin(n\pi x)/n\pi$
0	$-\cos(n\pi x)/n^2\pi^2$

Sine:

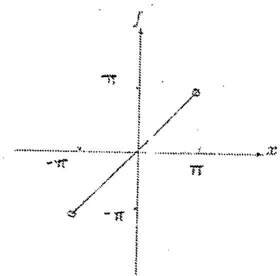
$\cancel{b_n = \frac{2}{n\pi}(\ldots - (-1)^n)} \Rightarrow f_s(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi}(-1)^{n+1} \cancel{\frac{2}{n\pi}(\ldots(-1)^n)} \sin(n\pi x)$

$b_n = -\frac{2}{n\pi}(-1)^n$

u	dv
x	$\sin(n\pi x)$
1	$-\cos(n\pi x)/n\pi$
0	$-\sin(n\pi x)/n^2\pi^2$

4. (10 Points) Let,

u	dv
x	e^{-inx}
1	$-\frac{1}{in}e^{-inx}$
0	$-\frac{1}{n^2}e^{-inx}$



be the graph of a periodic function on its principle domain. Find the complex Fourier series representation of f **and** convert this to an associated real Fourier series.

$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} x e^{-inx}\,dx = \frac{1}{2\pi}\left[\frac{-\pi}{in}e^{-in\pi} - \frac{\pi}{in}e^{in\pi}\right] = \frac{i}{n}(-1)^n,\ n \neq 0$

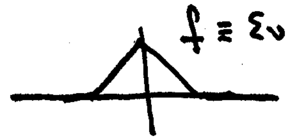
$C_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x\,dx = 0$

$f(x) = \sum_{n=1}^{\infty} \frac{i(-1)^n}{n} e^{inx} - \sum_{n=1}^{\infty} \frac{i(-1)^n}{n} e^{-inx} = \sum_{n=1}^{\infty} \frac{i(-1)^n}{n}\left[e^{inx} - e^{-inx}\right] =$

$= \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n}\sin(nx)$

5. (10 Points) Suppose that f is given as,

$$f(x) = \begin{cases} x+1, & -1 \le x < 0 \\ 1-x, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases} .$$



$f \equiv$ Even

Calculate the complex Fourier transform of f.

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx = \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} f(x) \cos(\omega x)\, dx =$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{1} (1-x) \cos(\omega x) dx =$$

u	dv
$1-x$	$\cos(\omega x)$
-1	$+\sin(\omega x)/\omega$
0	$-\cos(\omega x)/\omega^2$

$$= \sqrt{\frac{2}{\pi}} \left(-\frac{\cos(\omega)}{\omega^2} + \frac{1}{\omega^2} \right)$$

6. (Extra Credit)

$2y = \omega \Rightarrow$ ~~$\omega = y/2$~~ $y = \omega/2$

(a) Noting the identity, $2\sin^2(y) = 1 - \cos(2y)$, simplify the previous complex Fourier transform as much as possible.

$$\hat{f}(\omega) = \sqrt{\frac{2}{\pi}} \frac{1}{\omega^2} \left(2\sin^2(\omega/2) \right) = 2\sqrt{\frac{2}{\pi}} \cdot \frac{1}{4} \cdot \frac{\sin^2(\omega/2)}{(\omega/2)^2} = \frac{1}{\sqrt{2\pi}} [\mathrm{sinc}(\omega/2)]^2$$

(b) Let $\hat{f}(\omega) = \sum_{n=-\infty}^{\infty} \sqrt{2\pi} c_n \delta(\omega - n)$. Find the inverse Fourier transform of $\hat{f}$.

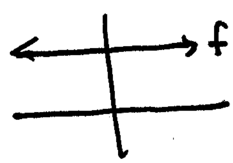
$$\mathcal{F}^{-1}\{\hat{f}\} = \sum_{n=-\infty}^{\infty} c_n \int_{-\infty}^{\infty} \delta(\omega - n) e^{+i\omega x} d\omega = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

(c) Suppose that $c_n = \dfrac{e^{in\pi} - e^{-in\pi}}{n}$ for $n \neq 0$ and $c_0 = \pi$. Graph the function $f(x) = \mathfrak{F}^{-1}\left\{\hat{f}\right\}$.

$$c_n = \frac{(-1)^n - (-1)^n}{n} = 0, \quad c_0 = \pi$$

$\Rightarrow$

$$f(x) = \mathcal{F}^{-1}\{\hat{f}\} = \pi \quad \Rightarrow$$



$f = \pi$